

# Penalty approximation and analytical characterization of the problem of super-replication under portfolio constraints

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## Abstract

In this paper, we consider the problem of super-replication under portfolio constraints in a Markov framework. More specifically, we assume that the portfolio is restricted to lie in a convex subset, and we show that the super-replication value is the smallest function which lies above the Black-Scholes price function and which is stable for the so-called face lifting operator. A natural approach to this problem is the penalty approximation, which not only provides a constructive smooth approximation, but also a way to proceed analytically .

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# 1 Introduction

The problem of super-replication under portfolio constraints has attracted a lot of interest. A precise formulation of this problem is provided in Section 2. Let us just observe that it consists in a non-standard stochastic control problem, with value defined as the minimal initial capital which requires to hedge some given contingent claim without risk.

The classical approach in the mathematical finance literature was to convert this problem into a standard stochastic control formulation by duality. This leads to many interesting developments in the field of stochastic processes, see e.g. Karatzas and Shreve [7] for an overview. However, this involves very technical probabilistic arguments, that we intend to avoid in the present work.

In a Markov framework, this problem can be approached by the classical dynamic programming technique. However, because of the constraints, we cannot expect to have a smooth solution of the the associated Bellman equation. In the previous literature, this problem is solved using the viscosity theory either on the dual formulation, or on the initial formulation by means of an original dynamic programming principle, see [9].

On the other hand, a natural approach to this problem is the penalty approximation, which not only provides a constructive smooth approximation, but also a way to proceed analytically. More specifically, we assume that the portfolio is restricted to lie in a convex subset, and we show that the super-replication value can be characterized in several ways, as the limit of the penalty approximations which are smooth, as the viscosity solution of the Bellman equation, and also as the smallest function which lies above the Black-Scholes price function and which is stable for the face lifting operator introduced in Broadie et al. [4]. An important feature of our analysis is that it does not require to pass by the dual formulation.

## 2 Problem formulation

### 2.1 The financial market

Let  $T > 0$  be a finite time horizon, and consider a complete probability space  $(\Omega, \mathcal{F}, \bar{P})$  equipped with a standard Brownian motion  $\bar{B} = \{(\bar{B}_t^1, \dots, \bar{B}_t^d), 0 \leq t \leq T\}$  valued in  $\mathbb{R}^d$ , and generating the ( $\bar{P}$ -augmented) filtration  $\mathbb{F}$ . We denote by  $\ell$  the Lebesgue measure on  $[0, T]$ .

The financial market consists of a non-risky asset  $S^0$  normalized to unity, i.e.  $S^0 \equiv 1$ , and  $d$  risky assets with price process  $S = (S^1, \dots, S^d)$  whose dynamics is defined by the stochastic differential equation:

$$S_0^i = s^i, \quad dS_t^i = S_t^i \left[ \mu^i(S_t) dt + \sum_{j=1}^d \sigma^{ij}(S_t) d\bar{B}_t^j \right]. \quad (2.1)$$

The functions  $\mu : \mathbb{R}_+^d \rightarrow \mathbb{R}^d$ , and  $\sigma : \mathbb{R}_+^d \rightarrow \mathcal{S}_{\mathbb{R}}(d)$  satisfy the Lipschitz condition:

$$|\text{diag}[s]\mu(s) - \text{diag}[s']\mu(s')| + |\text{diag}[s]\sigma(s) - \text{diag}[s']\sigma(s')| \leq K|s - s'|.$$

Under this condition, it is well-known that the SDE (2.1) has a unique strong solution. Moreover the coefficients  $\mu$  and  $\sigma$  are bounded.

$$\sup_{s \in (0, \infty)^d} |\mu(s)| + |\sigma(s)| < \infty.$$

*Remark 2.1.* The normalization of the non-risky asset to unity is, as usual, obtained by discounting, i.e., taking the non-risky asset as a *numéraire*.  $\square$

The matrix valued function  $\sigma$  is the *volatility* of the risky assets prices. We shall assume throughout this paper that the matrix  $\sigma(s)$  is invertible for every  $s \in (0, \infty)^d$ . We then set

$$\bar{\lambda}(s) := \sigma(s)^{-1}\mu(s), \quad \forall s \in (0, \infty)^d,$$

and we define the exponential local martingale

$$\begin{aligned}\bar{Z}_t &:= \mathcal{E} \left( - \int_0^t \bar{\lambda}(S_r) \cdot dW_r \right) \\ &:= \exp \left( - \int_0^t \bar{\lambda}(S_r) \cdot dW_r - \frac{1}{2} \int_0^t |\bar{\lambda}(S_r)|^2 dr \right).\end{aligned}$$

Next, we assume that  $\sigma^{-1}$  is bounded or more general, that the function  $\bar{\lambda}$  is such that

$$E [\bar{Z}_T] = 1, \quad (2.2)$$

i.e.,  $\bar{Z}$  is a martingale. Thus, if we denote by  $P$  the probability measure equivalent to  $\bar{P}$  induced by  $\bar{Z}$

$$P(A) := \bar{E} [\bar{Z}_t \mathbf{1}_A] \quad \forall A \in \mathcal{F}_t, \quad 0 \leq t \leq T,$$

where  $\bar{E}$  is the expectation operator under  $\bar{P}$ , then (by Girsanov Theorem), the process

$$B_t := \bar{B}_t + \int_0^t \bar{\lambda}(S_t) dt, \quad 0 \leq t \leq T,$$

is a standard Brownian motion under  $P$ , and the SDE (2.1) can be re-written in terms of  $B$

$$S_0^i = s^i, \quad dS_t^i = S_t^i \sum_{j=1}^d \sigma^{ij}(S_t) dB_t^j, \quad (2.3)$$

for every  $i = 1, \dots, d$ , in the filtered probability space  $(\Omega, P, \mathbb{F})$ .

## 2.2 Portfolio and wealth process

Let  $W_t$  denote the wealth at time  $t$  of some investor on the financial market. We assume that the investor allocates continuously his wealth between the non-risky asset and the risky assets. We shall denote by  $\pi_t^i$  the proportion of wealth invested in the  $i$ -th risky asset. This means that

$\pi_t^i W_t$  is the amount invested at time  $t$  in the  $i$ -th risky asset,

The remaining proportion of wealth  $1 - \sum_{i=1}^d \pi_t^i$  is invested in the non-risky asset.

An  $\mathbb{R}^d$ -valued process  $\pi$  is called an investment strategy if it is  $\mathbb{F}$ -adapted and satisfies the integrability condition

$$\int_0^T |\sigma(S_t)' \pi_t|^2 dt < \infty \quad P\text{-a.s.}$$

where primes denote transposition. We denote by  $\mathcal{A}$  the set of all investment strategies. Note that if  $c(t, s)$  is a  $\mathbb{R}^d$ -valued Borel measurable function  $c(t, s)$  satisfying  $|c(t, s)| \leq C(1 + |s|)$  for every  $s$ , then a feedback form  $\pi_t = c(t, S_t)$  is an investment strategy.

Under the so-called *self-financing condition* (i.e., the variation of the wealth process is only affected by the variation of the price process), every investment strategy  $\pi$  induces the following dynamics for the wealth process:

$$dW_t = W_t \pi_t \cdot \sigma(S_t) dB_t. \quad (2.4)$$

Observe that the above equation has a well-defined solution for every pair  $(w, \pi)$  of initial capital and investment strategy :

$$W_t^{w, \pi} := w \mathcal{E} \left( \int_0^t \pi_r \cdot \sigma(S_r) dB_r \right), \quad 0 \leq t \leq T.$$

Note that  $W^{w, \pi}$  is a super-martingale and a non-negative local martingale under  $P$ , for every  $(w, \pi)$  in  $\mathbb{R}_+ \times \mathcal{A}$ .

### 2.3 The hedging problem

Let  $K$  be a closed convex subset of  $\mathbb{R}^d$  containing the origin, and define the set of constrained strategies :

$$\mathcal{A}_K := \{ \pi \in \mathcal{A} : \pi \in K, \ell \otimes P\text{-a.s.} \}.$$

The set  $K$  represents some constraints on the investment strategies.

*Example 2.2 (Incomplete market).* Take  $K = \{x \in \mathbb{R}^d : x^i = 0\}$ , for some integer  $1 \leq i \leq d$ , means that trading on the  $i$ -th risky asset is forbidden.  $\square$

*Example 2.3 (No short-selling constraint).* Take  $K = \{x \in \mathbb{R}^d : x^i \geq 0\}$ , for some integer  $1 \leq i \leq d$ , means that the financial market does not allow to sell short the  $i$ -th asset.  $\square$

*Example 2.4 (No borrowing constraint).* Take  $K = \{x \in \mathbb{R}^d : x^1 + \dots + x^d \leq 1\}$  means that the financial market does not allow to sell short the non-risky asset or, in other words, borrowing from the bank is not available.  $\square$

In order to simplify the analysis, we shall assume that

$$K \quad \text{has non-empty interior.} \quad (2.5)$$

*Remark 2.5.* Condition (2.5) excludes important cases as the incomplete market of Example 2.2. In this particular example, it can be shown that the value function  $V(t, s)$  does not depend the  $s^i$ -variable, and the problem is treated following the analysis of this paper in the financial market with risky assets  $S^j$ ,  $j \neq i$ . The extension of this methodology to a general closed convex set  $K$  with empty interior can be found in [9].  $\square$

We next introduce a function  $g : [0, \infty) \rightarrow \mathbb{R}$ , and we assume that

$$g \quad \text{is non-negative, Lipschitz-continuous,} \quad (2.6)$$

and

$$g(s) \leq b(s) := C(1 + s^\gamma) = C \left( 1 + \prod_{i \leq d} (s^i)^{\gamma^i} \right), \quad (2.7)$$

for some constants  $C > 0$  and  $\gamma$  in  $K$ . The random variable

$$G := g(S_T)$$

is a European contingent claim. The primary goal of this paper is to study the following stochastic control problem

$$V(0, S_0) := \inf \{w \in \mathbb{R} : W_T^{w, \pi} \geq G, P\text{-a.s. for some } \pi \in \mathcal{A}_K\}, \quad (2.8)$$

i.e., the minimal initial capital which allows the seller of the contingent claim  $G$  to face, without risk, the payment  $G$  at time  $T$ , by means of some clever investment strategy on the financial market.

As usual, we shall denote by  $V(t, s)$  the dynamic version of the problem (2.8) which consists in the above super-replication problem started at the time origin  $t$  with initial data  $S_t = s$ .

### 3 The main results

Our main purpose is to obtain an analytical characterization of the value function of the super-replication problem (2.8).

We first provide a characterization of  $V$  by means of the associated Hamilton-Jacobi-Bellman equation. Denoting by  $V^0$  the value of  $V$  in the unconstrained case, we next show that  $V$  is the smallest function majorizing  $V^0$  and stable for some suitable non-linear operator.

Throughout this paper, we shall make use of the support function of the  $K$ , i.e.,

$$\delta(y) := \sup_{x \in K} x \cdot y \quad \text{for all } y \in \mathbb{R}^d,$$

and we denote by

$$\tilde{K} := \{y \in \mathbb{R}^n : \delta(y) < \infty\}$$

its effective domain, which is a closed convex cone containing the origin. Note that  $\delta : \mathbb{R}^d \rightarrow [0, \infty]$  is a lower semicontinuous and convex function, because  $0$  belongs to  $K$ , the function  $\delta$  is positively homogeneous,  $\delta(0) = 0$ , and

$$x \in K \Leftrightarrow \delta(y) - x \cdot y \geq 0, \quad \forall y \in \tilde{K}, |y| = 1,$$

and the restriction  $|y| = 1$  may be removed in view of the homogeneity of  $\delta$ .

Moreover, we need to define the following *hat* operator, as introduced by Brodie et al. [4]. For any function  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we define the function

$\hat{h} : \mathbb{R}_+ \longrightarrow \mathbb{R} \cup \{+\infty\}$  by

$$\hat{h}(s) := \sup_{y \in \tilde{K}} h(se^y) e^{-\delta(y)} \quad \forall s \in (0, \infty)^d, \quad (3.1)$$

with  $se^y = (s_1 e^{y_1}, \dots, s_d e^{y_d})$ . Note that always  $\hat{h} \geq h$ , and if  $h$  is differentiable at  $s$  and satisfies  $\hat{h}(s) = h(s) > 0$  then  $\delta(y)h(s) - y \cdot \text{diag}[s]Dh(s) \geq 0$  for every  $y$  in  $\tilde{K}$ , or equivalently,  $\text{diag}[s]Dh(s)/h(s)$  belongs to  $K$ .

### 3.1 The Hamilton-Jacobi-Bellman equation

Consider the non-linear parabolic PDE:

$$\begin{cases} \min \{-\mathcal{L}v(t, s), H(v(t, s), \text{diag}[s]Dv(t, s))\} = 0, \\ \text{for every } (t, s) \in [0, T) \times (0, \infty)^d, \end{cases} \quad (3.2)$$

where

$$\mathcal{L}v := \frac{\partial v}{\partial t}(t, s) + \frac{1}{2} \text{Tr} [\text{diag}[s]\sigma(t, s)\sigma(t, s)'\text{diag}[s]D^2v(t, s)],$$

and

$$H(r, p) := \inf \left\{ \delta(y)r - y \cdot p : y \in \tilde{K} \text{ and } |y| = 1 \right\},$$

where  $\sigma(t, s)'$  is the transposed matrix of  $\sigma(t, s)$ . In term of the so-called log-variables  $x = \ln s$  the HJB equation (3.2) becomes

$$\begin{cases} \min \left\{ -\tilde{\mathcal{L}}\tilde{v}(t, x), H(\tilde{v}(t, x), D\tilde{v}(t, x)) \right\} = 0, \\ \text{for every } (t, x) \in [0, T) \times \mathbb{R}^d, \end{cases} \quad (3.3)$$

with  $\tilde{v}(t, x) := v(t, s)$ ,  $\tilde{\sigma}(t, x) := \sigma(t, s)$ ,

$$\begin{aligned} \tilde{\mathcal{L}}\tilde{v}(t, x) := & \frac{\partial \tilde{v}}{\partial t}(t, x) - \frac{1}{2} \text{Tr} [\tilde{\sigma}(t, x)\tilde{\sigma}(t, x)'\text{diag}[D\tilde{v}(t, x)]] + \\ & + \frac{1}{2} \text{Tr} [\tilde{\sigma}(t, x)\tilde{\sigma}(t, x)'D^2\tilde{v}(t, x)], \end{aligned}$$

and  $x^i = \ln s^i$ , for  $i = 1, \dots, d$ . Note that (3.3) is non degenerate in  $\mathbb{R}^d$ , but the terminal data would have an exponential growth in  $x$ .

The importance of this operator  $H$  is highlighted by the following result.



**Lemma 3.1.** *Let  $v$  be a smooth function.*

(i) *If  $v > 0$  and  $H(v(t, s), \text{diag}[s]Dv(t, s)) \geq 0$ , then  $\hat{v} = v$ .*

(ii) *If  $-\mathcal{L}v \geq 0$  and  $v(T, \cdot) \geq g$ , then  $v \geq V$ .*

*Proof.* In the view of the discussion following the definition of the hat operator (3.1), we only have to show item (ii). By Itô's formula, we have

$$\begin{aligned} v(\tau, S_\tau^{t,s}) &= v(t, s) + \int_t^\tau \mathcal{L}v(r, S_r^{t,s}) + \int_t^\tau (\text{diag}[s]Dv \cdot \sigma)(r, S_r^{t,s})dB_r \\ &\leq v(t, s) + \int_t^\tau (\text{diag}[s]Dv \cdot \sigma)(r, S_r^{t,s})dB_r = W_\tau^{t,w,\pi}, \end{aligned}$$

where

$$\pi_r := (\text{diag}[s]v^{-1}Dv)(r, S_r^{t,s}) \quad \text{and} \quad w := v(t, s).$$

Since  $v$  is positive, this shows that  $\pi \in \mathcal{A}$ . Moreover, it follows from the fact that  $\hat{v} = v$  that  $\pi$  is valued in  $K$ . Hence,  $\pi \in \mathcal{A}_K$ . For  $\tau = T$ , the above inequality shows that

$$g(S_T^{t,s}) \leq v(T, S_T^{t,s}) \leq W_T^{t,w,\pi},$$

and therefore  $v(t, s) \geq V(t, s)$  by definition of the value function  $V$ .  $\square$

We next derive a smooth approximation of  $V$  by considering the non-linear parabolic PDE

$$-\mathcal{L}v(t, s) - \frac{1}{\varepsilon} H^-(v(t, s), \text{diag}[s]Dv(t, s)) = 0, \quad (3.4)$$

where  $H^- := \max\{0, -H\}$ .

**Theorem 3.2.** *Let condition (2.7) hold. Then, for every parameter  $\varepsilon > 0$ , there is a unique classic solution  $U^\varepsilon$  to the equation (3.4) satisfying the boundary condition*

$$U^\varepsilon(T, s) = g(s), \quad (3.5)$$

together with the growth condition

$$\sup_{(t,s) \in [0,T] \times \mathbb{R}_+^d} \frac{U^\varepsilon(t,s)}{1+s^\gamma} < \infty. \quad (3.6)$$

Moreover  $U^\varepsilon \leq V$  for every  $\varepsilon > 0$ , and the family  $(U^\varepsilon)_\varepsilon$  is a non-decreasing in  $\varepsilon$ .  $\square$

The proof of this result is given in Section 4.1. In view of the monotonicity of the family  $(U^\varepsilon)$ , we introduce the function

$$U(t,s) := \lim_{\varepsilon \searrow 0} U^\varepsilon(t,s) = \sup_{\varepsilon > 0} U^\varepsilon(t,s), \quad \forall (t,s) \in [0,T] \times (0,\infty)^d,$$

which is finite whenever  $V$  is finite. In the next statement, we use

$$V^*(t,s) := \limsup_{(t',s') \rightarrow (t,s)} V(t,s) \quad \text{and} \quad \underline{U}(t,s) := \liminf_{(\varepsilon,t',s') \rightarrow (0,t,s)} U^\varepsilon(t',s').$$

Observe that  $V^*$  and  $\underline{U}$  are finite whenever  $V$  is locally bounded.

**Theorem 3.3.** *Assume that  $V$  is locally bounded. Then:*

- (i)  $V^*$  is a viscosity sub-solution of (3.2), and  $V^*(T,s) \leq \hat{g}(s)$ .
- (ii)  $\underline{U}$  is a viscosity super-solution of (3.2), and  $\underline{U}(T,s) \geq \hat{g}(s)$ .  $\square$

The proof of this result is delayed until Sections 4.2 and 5.

*Remark 3.4.* For later use, we observe that

$$\underline{U}(t,s) \leq U_*(t,s) := \liminf_{(t',s') \rightarrow (t,s)} U(t',s').$$

To see this, let  $(\varepsilon_n, t_n, s_n)_n$  be a sequence such that  $\varepsilon_n \rightarrow 0$ ,  $(t_n, s_n) \rightarrow (t, s)$ , and  $U(t_n, s_n) \rightarrow U_*(t, s)$ . Then

$$\begin{aligned} \underline{U}(t,s) &= \liminf_{(\varepsilon,t',s') \rightarrow (0,t,s)} U^\varepsilon(t',s') \leq \liminf_{n \rightarrow \infty} U^\varepsilon(t_n, s_n) \leq \\ &\leq \liminf_{n \rightarrow \infty} U(t_n, s_n) = U_*(t,s), \end{aligned}$$

where we used the fact that  $U^\varepsilon \leq U$ .  $\square$

### 3.2 The case of a constant volatility matrix

We isolate the case of a constant volatility matrix as the corresponding value function can be characterized easier and under weaker assumption than in the general case, see Corollary 3.7. This case was studied in Broadie et al. [4] and [2]. We obtain here their results as a consequence of Theorem 3.3 (ii).

**Theorem 3.5.** *Let  $\sigma$  be a constant matrix, and assume that the payoff function  $g$  satisfies condition (2.7). Then  $V(t, s) = E[\hat{g}(S_T^{t,s})]$ .*

*Proof.* From Theorem 3.3 (ii), we deduce that  $-\mathcal{L}\underline{U} \geq 0$  and  $\underline{U}(T, \cdot) \geq \hat{g}$ . Observe that the function  $w(t, s) := E[\hat{g}(S_T^{t,s})]$  is a solution of the above (linear) PDE, i.e.  $-\mathcal{L}w = 0$  and  $w(T, \cdot) = \hat{g}$ . Then, it follows from (2.7) together with the maximum principle that  $\underline{U}(t, s) \geq E[\hat{g}(S_T^{t,s})]$ , and therefore  $V(t, s) \geq E[\hat{g}(S_T^{t,s})]$  by Theorem 3.2. The reverse inequality follows by applying Lemma 3.1 to the function  $w$ .  $\square$

### 3.3 Uniqueness and viscosity characterization

When the volatility matrix  $\sigma$  is not constant, the result of Theorem 3.5 does not hold. In order to characterize the value function  $V$  by means of the associated HJB equation, Theorem 3.3 has to be complemented by a uniqueness result.

**Theorem 3.6.** *Let  $u$  (resp.  $v$ ) be an upper semi-continuous (resp. lower semi-continuous) sub-solution (resp. super-solution) of the equation (3.2) on  $[0, T) \times (0, \infty)^d$  with  $u(T, \cdot) \leq \hat{g} \leq v(T, \cdot)$ , and*

$$\sup_{(t,s) \in [0,T] \times \mathbb{R}_+^d} \frac{|u(t,s)| + |v(t,s)|}{1 + s^\beta} < \infty \text{ for some } \beta \in \text{int}(K \cap \mathbb{R}_+^d).$$

*Assume further that either one of the following conditions*

$$\begin{cases} u \leq v \text{ on } [0, T] \times \partial\mathbb{R}_+^d, \text{ or} \\ \text{(H}_\mathbf{K}) & K \cap \text{int}(\mathbb{R}_+^d) \neq \emptyset, \end{cases}$$

*holds. Then  $u \leq v$  on  $[0, T] \times \mathbb{R}_+^d$ .*  $\square$

Here  $\mathbb{R}_+^d = [0, \infty)^d$  and  $\mathbb{R}_-^d = (-\infty, 0]^d$ . The proof of this result is given in Section 6. We now have all the ingredients for the characterization of the value function  $V$  by means of the associated HJB equation.

**Corollary 3.7.** *Let  $\gamma$  be in the interior  $\text{int}(K \cap \mathbb{R}_+^d)$ . Assume further that conditions  $(H_K)$  and (2.7) hold true. Then, the value function  $V$  is continuous on  $[0, T) \times \mathbb{R}_+^d$ ,  $V = U$ , and it is the unique viscosity solution of the equation (3.2) satisfying the boundary condition  $\lim_{t \nearrow T} V(t, s) = \hat{g}(s)$  together with the growth condition  $V(t, s) \leq C(1 + s^\gamma)$ , with a constant  $C$ .*

*Proof. 1.-* We first check that  $V(t, s) \leq v(t, s) := e^{\tau(t-T)}b(s)$  for sufficiently large  $\tau$ . Indeed, since  $\gamma \in K$  and  $\sigma(\cdot)$  is bounded, it is immediately checked that  $\hat{v} = v$ , and  $-\mathcal{L}v \geq 0$  for large  $\tau$ . Since  $g \leq b = v(T, \cdot)$ , we deduce from Lemma 3.1 that  $V \leq v$ .

*2.-* By Theorem 3.3, the functions  $V^*$  and  $\underline{U}$  are respectively super-solution and sub-solution of the equation (3.2) on  $[0, T) \times (0, \infty)^d$ , with  $V^*(T, \cdot) \leq \hat{g}$  and  $\underline{U}(T, \cdot) \geq \hat{g}$ . Moreover,  $V^*$  and  $\underline{U}$  inherit from  $V$  the growth conditions derived in the first step of this proof, recall that  $U \leq V$  by Theorem 3.2. We are then in the context of Theorem 3.6, and we can conclude that  $V^* \leq \underline{U}$ . Since  $V \geq U$  and  $U_* \geq \underline{U}$  by Remark 3.4, this provides the required result.  $\square$

*Remark 3.8.* Condition  $(H_K)$  excludes some important cases as the no short selling constraint of Example 2.3. It is possible to deal with such cases by analyzing the value function in the neighborhood of  $\partial\mathbb{R}_+^d$ . For instance, let  $d = 1$ , and suppose that

$$(H_g) \quad g \leq f \quad \text{for some convex function } f \text{ with } \hat{f} = f \text{ and } f(0) = g(0),$$

and let  $a$  be some constant satisfying  $\sigma(s)^2 \leq a$ . Then

(i)  $V(t, s) \leq F(t, s)$ , where  $F$  be the unique classical solution of the linear parabolic PDE

$$-F_t(t, s) - \frac{1}{2} a s^2 D^2 F(t, s) = 0 \quad \text{on } [0, T) \times (0, \infty) \quad (3.7)$$

with boundary condition

$$F(T, \cdot) = f. \quad (3.8)$$

To see this, observe that for all  $y \in \tilde{K}$ , the function

$$\delta(y)F(t, s) - y \cdot \text{diag}[s]DF(t, s)$$

is a classical super-solution of (3.7) with non-negative terminal condition, as a consequence of the fact that  $\hat{f} = f$ . This yields  $\hat{F} = F$ . Also, it is easily checked that  $F(t, \cdot)$  inherits the convexity of  $f$  for all  $t \in [0, T]$ . Then  $-\mathcal{L}F \geq 0$ , and we conclude that  $F \geq V$  by Lemma 3.1.

(ii) In particular, this implies that  $V^*(t, 0) \leq f(0) = g(0) = \hat{g}(0)$ . This is a consequence of the Feynman-Kac representation of the solution  $F$  of (3.7)–(3.8).

(iii) It is also easy to see that  $\underline{U}(t, 0) \geq g(0)$ . Indeed, since  $U_\varepsilon$  is a super-solution of the equation  $-\mathcal{L}U_\varepsilon \geq 0$  and  $U_\varepsilon(T, \cdot) = g$ , it follows from the classical maximum principle that  $U_\varepsilon(t, s) \geq E[g(S_T^{t,s})]$ . Hence, Fatou's lemma yields the desired result.

(iv) Since  $V^*(t, 0) \leq \underline{U}(t, 0)$  by (ii) and (iii), the statement of Corollary 3.7 holds by substituting  $(H_g)$  to  $(H_K)$ .  $\square$

### 3.4 An analytical characterization of $V$

The value function  $V$  was characterized in Corollary 3.7 by means of the notion of viscosity solutions. The following result provides an alternative probabilistic characterization, by working directly on the semigroup of conditional expectations associated to the process  $S$ . Notice that the statement of the following result does not appeal to any notion from PDE's, while the corresponding proof is based on the previous PDE-based developments.

We also observe that Condition  $(H_K)$  is required in the following statement, and that it can be weakened as in Remark 3.8.

**Theorem 3.9.** *Let  $\gamma$  be in the interior  $\text{int}(K \cap \mathbb{R}_+^d)$ , and assume that conditions  $(H_K)$  and (2.7) hold true. Then, the function  $V$  is the smallest Borel measurable function satisfying a growth condition as (2.7), i.e.,*

$$\sup_{(t,s) \in [0,T] \times \mathbb{R}_+^d} \frac{|v(t,s)|}{1+s^\gamma} < \infty,$$

and the following properties:

**P1**  $v(t,s) \geq E\{v(\theta, S_\theta^{t,s}) \mid S_t = s\}$  for all  $(t,s) \in [0,T) \times (0,\infty)^d$  and all stopping time  $\theta$  with values in  $[t,T]$ ,

**P2**  $\hat{v}_*(t, \cdot) = v_*(t, \cdot)$  for all  $t \in [0,T)$ ,

**P3**  $v_*(T, \cdot) \geq g$ , where  $v_*$  is the lower semicontinuous envelop, i.e.,

$$v_*(t,s) := \liminf_{(t',s') \rightarrow (t,s)} v(t',s'),$$

for every  $(t,s)$  in  $[0,T) \times (0,\infty)^d$ .

*Proof. 1.* We first check that  $V$  satisfies P1-P2-P3. From Corollary 3.7,  $V$  is continuous on  $[0,T) \times \mathbb{R}_+^d$ , and  $V(T-, \cdot) = \hat{g}$ . We then concentrate on P1 and P2.

1.1. Let  $\theta$  be a stopping time with values in  $[t,T]$ , and set

$$\theta_n := \theta \wedge \inf \{r > t : |\ln S_r^{t,s} - \ln s| > n\}.$$

For all  $\varepsilon > 0$ , the function  $U^\varepsilon$  is smooth by Proposition 3.2 and by means of Itô's formula we get

$$\begin{aligned} U^\varepsilon(t,s) - E[U^\varepsilon(\theta_n, S_{\theta_n}^{t,s})] &= \\ &= E \left[ \int_t^{\theta_n} -\mathcal{L}U^\varepsilon(r, S_r^{t,s}) dr - DU^\varepsilon(r, S_r^{t,s}) \cdot \text{diag}[S_r^{t,s}] \sigma(S_r^{t,s}) dB_r \right] \\ &= E \left[ \int_t^{\theta_n} -\mathcal{L}U^\varepsilon(r, S_r^{t,s}) dr \right] \\ &\geq \frac{1}{\varepsilon} E \left[ \int_t^{\theta_n} H^-(U^\varepsilon(r, S_r^{t,s}), \text{diag}[S_r] DU^\varepsilon(r, S_r^{t,s})) dr \right] \\ &\geq 0. \end{aligned}$$

Sending  $n$  to infinity, and using Fatou's lemma, we see that

$$U^\varepsilon(t, s) \geq E [U^\varepsilon(\theta, S_\theta^{t,s}) | S_t = s] .$$

We finally send  $\varepsilon$  to zero. Recalling that  $U^\varepsilon \nearrow V$  by Corollary 3.7, we deduce that  $V$  satisfies P1 by monotone convergence.

1.2. To see that  $V$  satisfies P2, we first observe that the inequality  $\hat{V} \geq V$  is always true by definition of the hat operator. We next introduce, for every fixed  $(t, s) \in [0, T] \times (0, \infty)^d$ , the family of continuous functions

$$h^{(y)}(r) := \ln [V(t, se^{ry})] - \delta(y)r, \quad \text{for } r \in \mathbb{R}^1 \text{ and } y \in \tilde{K} .$$

Since  $H(V, \text{diag}[s]DV) \geq 0$  in the viscosity sense, it follows that  $h^{(y)}$  is a viscosity super-solution of the equation  $-Dh^{(y)} \geq 0$ . Then  $h^{(y)}$  is non-increasing, and therefore  $h^{(y)}(0) \geq h^{(y)}(1)$ , i.e.  $\ln [V_*(t, s)] \geq \ln [V_*(t, se^y)] - \delta(y)$  for all  $y \in \tilde{K}$ . This provides

$$V(t, s) \geq V(t, se^y) e^{-\delta(y)}, \quad \forall y \in \tilde{K},$$

and therefore  $V(t, \cdot) \geq \hat{V}(t, \cdot)$ .

**2.** Now let  $v : [0, T] \times (0, \infty)^d \rightarrow \mathbb{R}$  be function satisfying P1-P2-P3 together with the growth condition (2.7). From P1, we deduce by classical techniques that  $v_*$  is a viscosity super-solution of  $-\mathcal{L}v_* \geq 0$ . From P2, we immediately show that  $v_*$  is a viscosity super-solution of  $H(v_*(s), \text{diag}[s]Dv_*(s)) \geq 0$ . Together with P3, we then have that  $v_*$  is a viscosity super-solution of (3.2) with  $v_*(T, \cdot) \geq \hat{g}$  and, under  $(H_g)$ ,  $v_*(t, 0) \geq g(0)$ . Since  $v$  satisfies the growth condition (2.7), we deduce from the comparison result of Theorem 3.6 that  $v_* \geq V$ . Since  $v \geq v_*$  by definition, this proves that  $v \geq V$ .  $\square$

## 4 The viscosity super-solution property

This section is devoted to the proof of the super-solution property of the function  $\underline{U}$  which was defined as the *relaxed semi-limit* of the family  $(U_\varepsilon)_\varepsilon$ .

## 4.1 Properties of the approximating family $(U^\varepsilon)_\varepsilon$

The proof of Theorem 3.2 is split in the following lemmas.

**Lemma 4.1.** *The PDE (3.4) has a unique classical solution  $U^\varepsilon$  satisfying the terminal condition (3.5) and the growth Condition (3.6). Moreover  $(U^\varepsilon)_\varepsilon$  is non-increasing in  $\varepsilon$ .*

*Proof.* By passing to the log-variables, as in (3.3), the equation (3.4) is reduced to a uniformly parabolic PDE with bounded coefficients, but with exponentially growth terminal conditions. Then, existence and uniqueness of a classical solution  $U^\varepsilon$  to the equation (3.4) satisfying the terminal condition  $U^\varepsilon(T, s) = g(s)$  together with the required growth condition (with constant depending on  $\varepsilon$ ) follows from classical results on uniformly parabolic equations, see Friedman [3], also [2], for details.

The monotonicity of the family  $(U^\varepsilon)_\varepsilon$  is a direct consequence of the maximum principle. Next, recall that the vector  $\gamma$  defining the bound  $b$  in (2.7) is in  $K$ . Then for a sufficiently large parameter  $\tau > 0$ , the function

$$f(t, s) := e^{-\tau(t-T)}b(s)$$

is a classical super-solution of the PDE (3.4)-(3.5) for any  $\varepsilon > 0$ , and the growth condition (3.6) follows from the classical maximum principle.  $\square$

**Lemma 4.2.** *For every  $\varepsilon > 0$ , we have  $U^\varepsilon \leq V$ .*

*Proof.* We shall concentrate on the case  $t < T$ , as the inequality holds obviously for  $t = T$ .

**1.** In order to prove this claim, we need to introduce some notations. Let  $\hat{y}(t, s)$  be such that

$$\begin{aligned} \delta(\hat{y}(t, s))U^\varepsilon(t, s) - \hat{y}(t, s) \cdot \text{diag}[s]DU^\varepsilon(t, s) &= \\ &= H^\varepsilon(U^\varepsilon(t, s), \text{diag}[s]DU^\varepsilon(t, s)), \end{aligned}$$

where

$$H^\varepsilon(r, p) := \min \left\{ \delta(y)r - y \cdot p : y \in \tilde{K} \text{ and } |y| \leq \varepsilon^{-1} \right\}.$$



Since  $\tilde{K}$  is a cone and  $\delta$  is positively homogeneous, it is easily checked that

$$H^\varepsilon(r, p) = -\frac{1}{\varepsilon}H^-(r, p) \quad (4.1)$$

One can clearly choose the function  $\hat{y}$  to be measurable so that the process

$$\nu_t := \hat{y}(t, S_t), \quad t \leq T,$$

is an  $\mathbb{F}$ -adapted process. We next define the probability measure  $P^\nu$  equivalent to  $P$  by its Radon-Nikodym density

$$\left. \frac{dP^\nu}{dP} \right|_{\mathcal{F}_T} = \exp \left( \int_0^T \nu_r \cdot dW_r - \frac{1}{2} \int_0^T |\nu_r|^2 dr \right)$$

and the real-valued process

$$X_t^{0,x,\pi} := x + \int_0^t \pi_r \cdot \sigma(S_r) dB_r - \frac{1}{2} \int_0^t |\pi_r \cdot \sigma(S_r)|^2 dr.$$

Since the process  $\nu$  is valued in  $\tilde{K}$ , it follows from a direct application of Itô's formula that the process

$$e^{-\int_0^t \delta(\nu_r) dr} X_t^{0,x,\pi} \quad \text{is a } P^\nu\text{-super-martingale} \quad (4.2)$$

for every  $\pi$  in  $\mathcal{A}_K$ .

**2.** Define the sequence of stopping times

$$\theta_n := T \wedge \inf \{ r > t : |\ln S_r^{t,s} - \ln s| \geq n \},$$

and observe that  $\theta_n \nearrow T$   $P$ -a.s. Since the process  $S^{t,s}$  is bounded up to the stopping time  $\theta_n$ , it follows from Itô's formula that

$$\begin{aligned} E^{P^\nu} \left[ e^{-\int_t^{\theta_n} \delta(\nu_r) dr} U^\varepsilon(\theta_n, S_{\theta_n}^{t,s}) \right] &= \\ &= U^\varepsilon(t, s) + E^{P^\nu} \left[ \int_t^{\theta_n} e^{-\int_t^r \delta(\nu_u) du} (\mathcal{L}U^\varepsilon + \varepsilon^{-1}h^\varepsilon)(r, S_r^{t,s}) dr \right], \end{aligned}$$

where  $h^\varepsilon(t, s) := H^\varepsilon(U^\varepsilon(t, s), \text{diag}[s]DU^\varepsilon(t, s))$ . By using (4.1), the PDE defining  $U^\varepsilon$  in Theorem 3.2 yields

$$\begin{aligned} U^\varepsilon(t, s) &= E^{P^\nu} \left[ e^{-\int_t^{\theta_n} \delta(\nu_r) dr} U^\varepsilon(\theta_n, S_{\theta_n}^{t,s}) \right] \\ &= \lim_{n \rightarrow \infty} E^{P^\nu} \left[ e^{-\int_t^{\theta_n} \delta(\nu_r) dr} U^\varepsilon(\theta_n, S_{\theta_n}^{t,s}) \right] \\ &= E^{P^\nu} \left[ e^{-\int_t^T \delta(\nu_r) dr} U^\varepsilon(T, S_T^{t,s}) \right]. \end{aligned} \quad (4.3)$$

Indeed, recalling that  $U^\varepsilon$  satisfies the growth condition (3.6), i.e.,

$$0 \leq U^\varepsilon(\theta_n, S_{\theta_n}^{t,s}) \leq C \left( 1 + \sup_{t \leq r \leq T} S_r^{t,s} \right) \in L^1$$

by means of classical estimates for stochastic differential equations (e.g., see Karatzas and Shreve [6]) the last equality (4.3) follows from the Lebesgue dominated convergence theorem.

**3.** We are now able for the proof of the inequality  $U^\varepsilon \leq V$  on  $[0, T) \times (0, \infty)^d$  by contradiction. Indeed, assume to the contrary that  $U^\varepsilon(t, s) - \eta > V(t, s)$  for some  $\varepsilon, \eta > 0$  and  $(t, s) \in [0, T) \times (0, \infty)^d$ , and let us work towards a contradiction. By definition of the super-replication problem (2.8), there exists an admissible portfolio  $\pi \in \mathcal{A}_K$  such that

$$X_T^{t, U^\varepsilon(t,s) - \eta, \pi} \geq g(S_T) = U^\varepsilon(T, S_T).$$

This inequality together with (4.2) and (4.3) imply

$$U^\varepsilon(t, s) - \eta \geq E^{P^\nu} \left[ e^{-\int_t^T \delta(\nu_r) dr} U^\varepsilon(T, S_T) \right] = U^\varepsilon(t, s),$$

which is the required contradiction.  $\square$

## 4.2 Asymptotic result for the family $(U_\varepsilon)_\varepsilon$

We now derive the viscosity super-solution property of the function  $\underline{U}$  which was defined from  $U^\varepsilon$  by sending  $\varepsilon$  to zero. This will be obtained by sending  $\varepsilon$  to zero in the PDE (3.4) satisfied by  $U^\varepsilon$ .

**Corollary 4.3.** *The function  $\underline{U}$  is a viscosity supersolution of the PDE (3.2) satisfying  $\underline{U}(T, s) \geq \hat{g}(s)$ .*

*Proof.* Let  $(t_0, s_0) \in [0, T) \times (0, \infty)^d$  and  $\varphi \in C^2([0, T) \times (0, \infty)^d, \mathbb{R})$  be such that

$$0 = (\underline{U} - \varphi)(t_0, s_0) = (\text{strict}) \min(\underline{U} - \varphi).$$

We have to prove that

$$-\mathcal{L}\varphi(t_0, s_0) \geq 0 \quad \text{and} \quad H(\varphi, \text{diag}[s]D\varphi)(t_0, s_0) \geq 0. \quad (4.4)$$

1. Let  $B$  be some open ball containing  $(t_0, s_0)$ . By definition of  $\underline{U}$ , there is a sequence  $(\varepsilon_n, t_n, s_n)_{n \geq 1}$  such that

$$(\varepsilon_n, t_n, s_n) \longrightarrow (0, t_0, s_0) \quad \text{and} \quad U^{\varepsilon_n}(t_n, s_n) \longrightarrow \underline{U}(t_0, s_0).$$

Let  $(\bar{t}_n, \bar{s}_n)$  be such that

$$(U^{\varepsilon_n} - \varphi)(\bar{t}_n, \bar{s}_n) = \min_{\bar{B}} (U^{\varepsilon_n} - \varphi),$$

where  $\bar{B}$  denotes the closure of  $B$ . In the following Step 2, we shall verify that the claim

$$(\bar{t}_n, \bar{s}_n) \longrightarrow (t_0, s_0) \quad (4.5)$$

holds. Then, for sufficiently large  $n$ ,  $(\bar{t}_n, \bar{s}_n)$  is an interior minimizer of the difference  $(U^{\varepsilon_n} - \varphi)$ , and it follows from Theorem 3.2 that

$$-\mathcal{L}\varphi(\bar{t}_n, \bar{s}_n) + \frac{1}{\varepsilon_n} [\delta(y)\varphi(\bar{t}_n, \bar{s}_n) - y \cdot \text{diag}[\bar{s}_n]D\varphi(\bar{t}_n, \bar{s}_n)]$$

for every  $y$  in  $\tilde{K}$  with  $|y| \leq 1$ . Hence, as  $n$  goes to infinity we deduce  $H(\varphi(t_0, s_0), \text{diag}[s_0]D\varphi(t_0, s_0)) \geq 0$ . The remaining inequality of (4.4) is obtained by setting  $y$  to zero and sending  $n$  to infinity.

2. It remains to prove (4.5). Since  $(\bar{t}_n, \bar{s}_n) \in \bar{B}$ , the sequence  $\{(\bar{t}_n, \bar{s}_n) : n \geq 1\}$  converges to some  $(\bar{t}, \bar{s})$  in  $\bar{B}$ , after possibly passing to a subsequence. We then compute that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} (U^{\varepsilon_n} - \varphi)(t_n, s_n) \geq \liminf_{n \rightarrow \infty} (U^{\varepsilon_n} - \varphi)(\bar{t}_n, \bar{s}_n) \geq \\ &\geq \liminf_{(\varepsilon, t', s') \rightarrow (0, \bar{t}, \bar{s})} U^\varepsilon(t', s') - \varphi(\bar{t}, \bar{s}) = (\underline{U} - \varphi)(\bar{t}, \bar{s}). \end{aligned}$$

Since  $(t_0, s_0)$  is a strict minimizer of the difference  $(\underline{U} - \varphi)$ , this proves that  $(\bar{t}, \bar{s}) = (t_0, s_0)$ .

3. Following the lines of the proof of Lemma 5.2 in Soner and Touzi [8], we see that  $H(\underline{V}(T, s), \text{diag}[s]D\underline{V}(T, s)) \geq 0$  in the viscosity sense. This technical part is nothing but a passage to the limit as  $t \nearrow T$  in the equation  $H(\underline{V}(t, s), \text{diag}[s]D\underline{V}(t, s)) \geq 0$ .

We then argue as in Step 1.2 of the proof Theorem 3.9 to see that  $\underline{U}(T, \cdot) \geq \hat{U}(T, \cdot)$ . On the other hand, it follows from the definition of the  $U^\varepsilon$ , the continuity of  $g$ , together with Fatou's lemma, that  $\underline{U}(T, \cdot) \geq g$ . Thus  $\underline{U}(T, \cdot) \geq \hat{g}$ .  $\square$

## 5 The viscosity sub-solution property

We first recall from [8] that the value function  $V$  satisfies the following geometric dynamic programming principle : for all stopping time  $\tau$  with values in  $[t, T]$ ,

$$V(t, s) = \inf \left\{ w : W_\tau^{t, w, \pi} \geq V(\tau, S_\tau^{t, s}) \text{ for some } \pi \in \mathcal{A}_K \right\}.$$

In this section, we shall make use of this result in order to prove that the value function of the problem (2.8) is a (discontinuous) viscosity sub-solution of (3.2).

Notice that the geometric dynamic programming principle is also suitable for proving the super-solution property of the value function. However, due to the unbounded nature of the control  $\pi$ , this derivation leads to heavy

technicalities, see e.g. [9]. In this paper, the super-solution property was derived in Section 4.2 by an alternative argument which produces, as a by-product, a smooth approximating sequence for the value function  $V$ .

Since we have no knowledge of the regularity of the value function  $V$ , we introduce the upper semi-continuous envelope of  $V$ ,

$$V^*(t, s) := \limsup_{(t', s') \rightarrow (t, s)} V(t', s').$$

**Proposition 5.1.** *Assume that  $V$  is locally bounded. Then  $V^*$  is a viscosity subsolution of (3.2) satisfying  $V^*(T, \cdot) \leq \hat{g}$ .*

*Proof. 1.-* We first show that  $V^*$  is a viscosity subsolution of (3.2). In order to simplify the presentation, we shall pass to the log-variables. Set  $x := \ln w$ ,  $X^{t,x,\pi} := \ln W^{t,w,\pi}$ , and  $v := \ln V$ . By Itô's formula, the controlled process  $X^{t,x,\pi}$  is given by

$$X_u^{t,x,\pi} = x + \int_t^u \pi_r \cdot \sigma(S_r) dB_r - \frac{1}{2} \int_t^u |\sigma(S_r)' \pi_r|^2 dr.$$

With this change of variable, Proposition 5.1 states that  $v^*$  satisfies on  $[0, T) \times (0, \infty)^d$  the equation

$$\min \left\{ -\hat{\mathcal{L}}v^*(t, s), \hat{H}(\text{diag}[s] Dv_s^*(t, s)) \right\} \leq 0$$

in the viscosity sense, where

$$\begin{aligned} \hat{\mathcal{L}}v^*(t, s) &:= v_t^*(t, s) + \\ &+ \frac{1}{2} \text{Tr} \left[ \text{diag}[s] \sigma(s) \sigma(s)' \text{diag}[s] (D^2v^* + Dv^*(Dv^*)')(t, s) \right], \end{aligned}$$

and

$$\hat{H}(p) := H(1, p).$$

We argue by contradiction. Let  $(t_0, s_0) \in [0, T) \times (0, \infty)^d$  and  $\varphi \in C^2([0, T) \times (0, \infty)^d)$  be such that

$$0 = (v^* - \varphi)(t_0, s_0) = (\text{strict}) \max(w^* - \varphi),$$

and suppose that

$$-\hat{\mathcal{L}}\varphi(t_0, s_0) > 0 \quad \text{and} \quad \hat{H}(\text{diag}[s_0]D\varphi(t_0, s_0)) > 0 .$$

Since  $K$  has non-empty interior by (2.5), the last condition is equivalent to

$$\text{diag}[s_0]D\varphi(t_0, s_0) \in \text{int}(K) .$$

Set  $\hat{p}(t, s) := \text{diag}[s]D\varphi(t, s)$ . Let  $0 < \alpha < T - t_0$  be an arbitrary scalar and define the neighborhood of  $(t_0, s_0)$

$$\mathcal{N} := \{(t, s) \in B_\alpha(t_0, s_0) : \hat{p}(t, s) \in K \text{ and } -\mathcal{L}^y\varphi(t, s) \geq 0\} ,$$

where

$$B_\alpha(t_0, s_0) = \{(t, s) : |t - t_0| + \sum_{1 \leq i \leq d} |\ln(s^i/s_0^i)| < \alpha\} .$$

Since  $(t_0, s_0)$  is a strict maximizer of  $(v^* - \varphi)$ , we can define

$$-3\beta := \max_{\partial\mathcal{N}}(v^* - \varphi) < 0 .$$

Let  $(t_1, s_1)$  be some element in  $\mathcal{N}$  such that

$$x_1 := v(t_1, s_1) \geq v^*(t_0, s_0) - \beta = \varphi(t_0, s_0) - \beta ,$$

and consider the controlled process

$$X^{t_1, x_1 - \beta, \hat{\pi}} = \ln W^{t_1, w_1 e^{-\beta}, \hat{\pi}} \quad \text{with control} \quad \hat{\pi}_t := \hat{p}(t, S_t^{t_1, s_1}) .$$

This defines a wealth process  $W^{t_1, w_1 e^{-\beta}, \hat{\pi}}$  at least up to the stopping time  $\hat{\theta}$  given by

$$\hat{\theta} := \inf \{r > t_0 : (r, S_r^{t_1, s_1}) \notin \mathcal{N}\} .$$

Now, it follows from the inequality  $v \leq v^* \leq \varphi - 3\beta$  on  $\partial\mathcal{N}$  that

$$\begin{aligned} X_{\hat{\theta}}^{t_1, x_1 - \beta, \hat{\pi}} - v(\hat{\theta}, S_{\hat{\theta}}^{t_1, s_1}) &\geq 2\beta + v(t_1, s_1) - \varphi(\hat{\theta}, S_{\hat{\theta}}^{t_1, s_1}) \geq \\ &\geq \beta + \varphi(t_1, s_1) - \varphi(\hat{\theta}, S_{\hat{\theta}}^{t_1, s_1}) . \end{aligned}$$

Applying Itô's formula to the smooth function  $\varphi$ , we get

$$\begin{aligned} X_{\hat{\theta}}^{t_1, x_1 - \beta, \hat{\pi}} - v(\hat{\theta}, S_{\hat{\theta}}^{t_1, s_1}) &\geq \beta + \int_{t_1}^{\hat{\theta}} \hat{\mathcal{L}}\varphi(t, S_r^{t_1, s_1}) dr \geq \\ &\geq \beta > 0 \quad P\text{-a.s.} \end{aligned}$$

where the diffusion term vanishes by definition of  $\hat{\pi}$ . This proves the inequality

$$W_{\hat{\theta}}^{t_1, w_1 e^{-\beta}, \hat{\pi}} > V(\hat{\theta}, S_{\hat{\theta}}^{t_1, s_1}),$$

which is in contradiction with the geometric dynamic programming principle stated in the beginning of this section.

**2.-** Following the lines of the proof of Lemma 5.3. in Soner and Touzi [8], we can pass to the limit as  $t \nearrow T$  in the subsolution property of the value function, and we deduce from the result of the previous step that

$$\min \{V^*(T, s) - g(s), H(\bar{V}^*(T, s), \text{diag}[s]DV^*(T, s))\} \leq 0 \quad (5.1)$$

in the viscosity sense. Clearly,  $\hat{g} \geq g$ . It is also easily checked from the definition of  $\hat{g}$  that  $H(\hat{g}(s), \text{diag}[s]D\hat{g}(s)) \geq 0$  in the viscosity sense. Hence  $\hat{g}$  is a viscosity super-solution of the equation appearing on the left hand-side of (5.1). By the comparison result reported in Theorem 4.3 of Barles [1], we conclude that  $\hat{g} \geq V^*(T, \cdot)$  (this is the only place where we need  $g$  to be Lipschitz), completing the proof of the proposition.  $\square$

## 6 Comparison result

The last ingredient which has been used in the proof of Corollary 3.7 is the comparison result of Theorem 3.6. The assumption that  $\gamma$  belongs to  $\text{int}(K)$ , for the given bound  $b(s) := C(1 + s^\gamma)$ , is a key condition in the quoted results. Thus, there exists a vector

$$\bar{\gamma} \in \text{int}(K) \quad \text{such that} \quad \bar{\gamma} - \gamma \in \text{int}(\mathbb{R}_+^d). \quad (6.1)$$

We next consider the following function whose definition depends on whether or not Condition  $(\mathbf{H}_K)$  is in force

$$\beta(t, s) := Ce^{\tau(T-t)} (1 + s^{\bar{\gamma}}), \quad (6.2)$$

$$\beta(t, s) := Ce^{\tau(t-T)} (1 + s^{\bar{\gamma}} + s^{\underline{\gamma}}), \quad (6.3)$$

for some  $\underline{\gamma}$  in  $\text{int}(K \cap \mathbb{R}_-^d) \neq \emptyset$  under  $(\mathbf{H}_K)$ . Here, the parameter  $\tau$  is chosen so that

$$-\mathcal{L}\beta \geq 0. \quad (6.4)$$

We also observe that, since  $0 \in K$  and  $\bar{\gamma}, \underline{\gamma} \in \text{int}(K)$ , we have

$$H(\beta(t, s), \text{diag}[s]D\beta(t, s)) > 0 \quad \text{for all } (t, s) \in [0, T] \times \mathbb{R}_+^d. \quad (6.5)$$

We are now ready to prove the comparison result.

*Proof of Theorem 3.6.* Let  $\lambda > 0$  be some given parameter, and define the functions

$$u^\lambda(t, s) := e^{\lambda t} u(t, s) \quad v^\lambda(t, s) := e^{\lambda t} v(t, s) \quad \text{and} \quad \beta^\lambda(t, s) := e^{\lambda t} \beta(t, s).$$

Then, with

$$\mathcal{L}^\lambda w := \mathcal{L}w - \lambda w,$$

the functions  $u^\lambda$  and  $v^\lambda$  are viscosity sub-solution and super-solution of the equation

$$\min \{ -\mathcal{L}^\lambda w(t, s), H(w(t, s), \text{diag}[s]Dw(t, s)) \} = 0, \quad \forall (t, s) \quad (6.6)$$

satisfying  $u^\lambda(T, \cdot) \leq v^\lambda(T, \cdot)$  and the growth condition stated in the proposition. Also, by means (6.4)-(6.5) of the preceding discussion, one deduces that  $\beta^\lambda$  satisfies

$$-\mathcal{L}^\lambda \beta^\lambda \geq 0 \quad \text{and} \quad H(\beta^\lambda(t, s), \text{diag}[s]D\beta^\lambda(t, s)) > 0. \quad (6.7)$$



In the rest of this proof, we drop the  $\lambda$  exponent in the notation of  $u^\lambda$ ,  $v^\lambda$ ,  $\beta^\lambda$ , and we simply write  $u$ ,  $v$ ,  $\beta$ . We shall also denote  $\mathcal{O} := [0, T] \times (0, \infty)^d$ .

Let  $\alpha > 0$  and  $\varepsilon > 0$  be two parameters and define

$$M_\alpha := \sup_{\mathcal{O} \times \mathcal{O}} \phi(t, s, t', s'), \quad (6.8)$$

where

$$\begin{aligned} \phi(t, s, t', s') &:= u(t, s) - v(t', s') - \frac{\alpha}{2} (|t - t'|^2 + |s - s'|^2) - \\ &\quad - \varepsilon (\beta(t, s) + \beta(t', s')). \end{aligned}$$

Assume to the contrary that  $u(t_0, s_0) - v(t_0, s_0) > 0$  for some  $(t_0, s_0) \in \mathcal{O}$ , so that

$$M_\alpha \geq (u - v)(t_0, s_0) - 2\varepsilon\beta(t_0, s_0) =: \eta > 0, \quad (6.9)$$

and let us work towards a contradiction.

**1.-** From the growth conditions assumed on  $u$  and  $v$ , together with (6.1), (6.2), (6.3), it follows that

$$M_\alpha = \phi(t_\alpha, s_\alpha, t'_\alpha, s'_\alpha) \quad (6.10)$$

for some  $(t_\alpha, s_\alpha), (t'_\alpha, s'_\alpha) \in \overline{\mathcal{O}}$ . We also estimate that  $M_\alpha \geq u(T, s_\alpha) - v(T, s_\alpha) - 2\varepsilon\beta(T, s_\alpha)$ , and therefore

$$\varepsilon [\beta(t_\alpha, s_\alpha) + \beta(t'_\alpha, s'_\alpha)] \leq C [1 + u(T, s_\alpha) - v(T, s_\alpha)].$$

From the bound  $(|u| + |v|)(T, s) \leq C(1 + s^\gamma)$  and (6.1), this implies that the families  $(t_\alpha, s_\alpha)_\alpha$  and  $(t'_\alpha, s'_\alpha)_\alpha$  are located in a compact subset of  $[0, T] \times [0, \infty)$  (the bounds depending on  $\varepsilon$ ). By Lemma 3.1 in Crandall et al. [5], we then conclude that when  $\alpha \rightarrow \infty$

$$(t_\alpha, s_\alpha) \longrightarrow (\bar{t}, \bar{s}) \in \overline{\mathcal{O}} \quad \text{and} \quad \alpha (|t_\alpha - t'_\alpha|^2 + |s_\alpha - s'_\alpha|^2) \longrightarrow 0 \quad (6.11)$$

after possibly passing to a subsequence.

**2.** In this step, we prove that (6.9) implies that  $\bar{t} < T$  and  $\bar{s} \in \text{int}(\mathbb{R}_+^d)$ .

2.1. If  $\bar{t} = T$ , we directly compute that

$$\lim_{\alpha \rightarrow \infty} M_\alpha = u(T, \bar{s}) - v(T, \bar{s}) - 2\varepsilon\beta(T, \bar{s}) \leq u(T, \bar{s}) - v(T, \bar{s}),$$

so that the assumption that  $u(T, \cdot) - v(T, \cdot) \leq 0$  is in contradiction with (6.9).

2.2. When  $u \leq v$  on  $[0, T] \times \partial\mathbb{R}_+^d$ , the case  $\bar{s} \in \partial\mathbb{R}_+^d$  leads to a contradiction by the same argument as above.

2.3. Under Condition  $(\mathbf{H}_K)$ , we also have that  $\bar{s} \in \text{int}(K)$ . This follows from the extra term  $s^2$  in the definition of the function  $\beta$ .

**3.-** From the previous step, we have that  $(t_\alpha, s_\alpha), (t'_\alpha, s'_\alpha)$  is a local maximizer in (6.8). By Theorem 3.2 (and the discussion thereafter) in Crandall et al. [5], there exist two symmetric matrices  $A$  and  $B$  such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad (6.12)$$

and

$$\begin{aligned} (p_\alpha + \varepsilon\beta_t(t_\alpha, s_\alpha), q_\alpha + \varepsilon D\beta(t_\alpha, s_\alpha), A + \varepsilon D^2\beta(t_\alpha, s_\alpha)) &\in \bar{J}^{2,+}u(t_\alpha, s_\alpha), \\ (p_\alpha - \varepsilon\beta_t(t'_\alpha, s'_\alpha), q_\alpha - \varepsilon D\beta(t'_\alpha, s'_\alpha), B - \varepsilon D^2\beta(t'_\alpha, s'_\alpha)) &\in \bar{J}^{2,-}v(t'_\alpha, s'_\alpha), \end{aligned}$$

where

$$p_\alpha := \alpha(t - t'), \quad q_\alpha := \alpha(s - s'),$$

and  $J^{2,+}w(z)$  and  $J^{2,-}w(z)$  denote the closed superjet and subjet of the function  $w$  at the point  $z$ , see [5] for the definitions. By the viscosity properties of the functions  $u$  and  $v$ , this implies that

$$\min \left\{ -\varepsilon\mathcal{L}^0\beta(t_\alpha, s_\alpha) + \lambda u(t_\alpha, s_\alpha) - p_\alpha - \frac{1}{2}\text{Tr}[a(s_\alpha)A], H_\alpha \right\} \leq 0, \quad (6.13)$$

$$\min \left\{ \varepsilon\mathcal{L}^0\beta(t'_\alpha, s'_\alpha) + \lambda v(t'_\alpha, s'_\alpha) - p_\alpha - \frac{1}{2}\text{Tr}[a(s'_\alpha)B], H'_\alpha \right\} \geq 0, \quad (6.14)$$

where  $a(s) := \text{diag}[s]\sigma(s)\sigma(s)'\text{diag}[s]$ ,

$$\begin{aligned} H_\alpha &:= H(u(t_\alpha, s_\alpha), \text{diag}[s_\alpha](q_\alpha + \varepsilon\beta(t_\alpha, s_\alpha))) , \\ H'_\alpha &:= H(v(t'_\alpha, s'_\alpha), \text{diag}[s'_\alpha](q_\alpha - \varepsilon\beta(t'_\alpha, s'_\alpha))) . \end{aligned}$$

4.- Since  $\{y \in \tilde{K} : |y| = 1\}$  is compact, it contains some  $y_\alpha$  such that

$$\begin{aligned} H_\alpha - H'_\alpha &= \delta(y_\alpha)u(t_\alpha, s_\alpha) - y_\alpha \cdot \text{diag}[s_\alpha](q_\alpha + \varepsilon\beta(t_\alpha, s_\alpha)) - H'_\alpha \\ &\geq \delta(y_\alpha)[u(t_\alpha, s_\alpha) - u(t'_\alpha, s'_\alpha)] - y_\alpha \cdot (\text{diag}[s_\alpha] - \text{diag}[s'_\alpha])q_\alpha \\ &\quad - \varepsilon y_\alpha [\text{diag}[s_\alpha]D\beta(t_\alpha, s_\alpha) + \text{diag}[s'_\alpha]D\beta(t'_\alpha, s'_\alpha)] \\ &\geq \delta(y_\alpha) \left[ \eta + \frac{\alpha}{2}(|t_\alpha - t'_\alpha|^2 + |s_\alpha - s'_\alpha|^2) \right] - \varepsilon y_\alpha (\text{diag}[s_\alpha] - \text{diag}[s'_\alpha])q_\alpha \\ &\quad + \varepsilon [\delta(y_\alpha)\beta - y_\alpha \cdot D\beta](t_\alpha, s_\alpha) + \varepsilon [\delta(y_\alpha)\beta - y_\alpha \cdot D\beta](t'_\alpha, s'_\alpha) \end{aligned}$$

We now send  $\alpha$  to infinity. By possibly passing to a subsequence, we obtain for some  $\bar{y} \in \tilde{K}$ ,  $|\bar{y}| = 1$

$$\liminf_{\alpha \rightarrow \infty} H_\alpha - H'_\alpha \geq \eta\delta(\bar{y}) + 2\varepsilon [\delta(\bar{y})\beta - \bar{y} \cdot D\beta](\bar{t}, \bar{s}) > 0, \quad (6.15)$$

where the last inequality follows from (6.7) together with the non-negativity of the support function  $\delta$ .

5.- Recall that  $-\mathcal{L}^\lambda\beta = \lambda\beta - \mathcal{L}\beta \geq 0$  by (6.7) (with the simplified notation  $\beta$  for  $\beta^\lambda$ ). It then follows from (6.13), (6.14), and (6.15) that for sufficiently large  $\alpha$

$$-\varepsilon\lambda\beta(t_\alpha, s_\alpha) + \lambda u(t_\alpha, s_\alpha) - p_\alpha - \frac{1}{2}\text{Tr}[a(s_\alpha)A] \leq 0,$$

and

$$\varepsilon\lambda\beta(t'_\alpha, s'_\alpha) + \lambda v(t'_\alpha, s'_\alpha) - p_\alpha - \frac{1}{2}\text{Tr}[a(s'_\alpha)B] \geq 0,$$

which implies that

$$\begin{aligned} 0 &\leq \varepsilon [\beta(t_\alpha, s_\alpha) + \beta(t'_\alpha, s'_\alpha)] + v(t'_\alpha, s'_\alpha) - u(t_\alpha, s_\alpha) + \frac{1}{2}\text{Tr}[a(s_\alpha)A - a(s'_\alpha)B] \\ &\leq \varepsilon [\beta(t_\alpha, s_\alpha) + \beta(t'_\alpha, s'_\alpha)] + v(t'_\alpha, s'_\alpha) - u(t_\alpha, s_\alpha) + C\alpha|s_\alpha - s'_\alpha|^2 \end{aligned}$$

for some constant  $C$ , where the last inequality follows from (6.12), see Example 3.6 in Crandall et al. [5]. By (6.9) and (6.10), this provides

$$0 \leq -\eta - C'\alpha (|t_\alpha - t'_\alpha|^2 + |s_\alpha - s'_\alpha|^2) .$$

Since  $\eta > 0$  is independent of  $\alpha$ , the above inequality is in contradiction with (6.11).  $\square$

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