Stochastic Games for N Players

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Dedicated to Professor D. LUENBERGER

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1 Introduction

The objective of this paper is to present some results concerning a class of stochastic games for N players. The method consists in applying the Dynamic Programming approach. The value functions are the solutions of a system of partial differential equations . From regularity results, one is able to construct continuous feedbacks, which will represent optimal controls for each player. The standard verification approach will show the optimality. The optimality is to be taken in the sense of Nash, [5], see also J.P. Aubin [1]. The general theory of regularity we rely on is detailed in the book of the authors, [2], and concerns a class of systems of nonlinear partial differential equations, which fits particularly well with stochastic games.

2 Statement of the Problem and Results

2.1 Description of a stochastic game

Let

(2. 1) $$\Omega = C^0([0,+\infty);R^n),\ \mathcal{A} = \text{ Borel } \sigma\text{−algebra on } \Omega$$

The elements of Ω are denoted by $\omega \equiv \omega(t)$, and we provide Ω with a probability law P such that

(2. 2) $\omega(t)$ is a standardized n dimensional Wiener process.

We then set

(2. 3) $$\mathcal{F}^t = \sigma\{\omega(s), s \leq t, \omega \in \Omega\}$$

A trajectory starting at x, is simply

(2. 4) $$x(t;\omega) = x + \omega(t)$$

We now consider N players, each of them acts through a control $v_\nu(t)$, $\nu = 1, \cdots, N$. We assume

(2. 5) $$v(t) = (v_1(t), \cdots, v_N(t)),\ \text{adapted process with bounded values in } R^{nN}$$

which we call an admissible control.

Let also

(2. 6) $$g(x),\ \text{measurable bounded function with values in } R^n$$

To a pair $x, v(t)$, where $v(t)$ is a control vector as above, we associate the process

(2. 7) $$\beta_{x,v}(t) = g(x(t)) + \sum_{\mu} v_{\mu}(t)$$

and the probability $P_{x,v}$ such that

(2. 8) $$\frac{dP_{x,v}}{dP}|_{\mathcal{F}^t} = \exp\{\int_0^t \beta_{x,v}(s)\, d\omega(s) - \frac{1}{2}\int_0^t |\beta_{x,v}(s)|^2\, ds\}.$$

From the Girsanov Theorem, if we introduce the process

(2. 9) $$w_{x,v}(t) = \omega(t) - \int_0^t \beta_{x,v}(s)\, ds$$

then the system $\Omega, \mathcal{A}, \mathcal{F}^t, P_{x,v}, w_{x,v}(t)$ forms a probability system in which $w_{x,v}(t)$ is an $\mathcal{F}^t$ standardized Wiener process. Note that from (2. 9) one has

(2. 10) $$dx = (g(x(t)) + \sum_{\mu} v_{\mu}(t))\, dt + dw_{x,v}(t), \quad x(0) = x$$

Let now

(2. 11) $$\mathcal{O} = \text{ open smooth bounded domain of } R^n$$

and let

(2. 12) $$\tau_x = \inf\{t | x(t) \notin \mathcal{O}\}$$

We shall stop the process $x(t)$ at the exit of the domain $\mathcal{O}$, and to save the notation, we shall still denote by $x(t)$ the stopped process. Let also

(2. 13) $$f_{\nu}(x), \text{ scalar measurable bounded function}$$

we set

(2. 14) $$J_{\nu}(x, v) = E_{x,v} \int_0^{\tau_x} e^{-ct}(f_{\nu}(x(t)) + \frac{1}{2}|v_{\nu}(t)|^2 + \theta v_{\nu}(t).\bar{v}_{\nu}(t))dt$$

with the notation

(2. 15) $$\bar{v}_{\nu} = \sum_{\mu \neq \nu} v_{\mu}$$

and where

(2. 16) $$c > 0, \theta \text{ a real parameter.}$$

We shall also use the notation

(2. 17) $$v = (v_{\nu}, \bar{v}^{\nu})$$

where , of course the part $\bar{v}^{\nu}$ represents all components which are different from v_{ν}.
A Nash point for the game defined by the functionals (2. 14) is a control $\hat{v}(.)$ such that

(2. 18) $$J_{\nu}(x, \hat{v}_{\nu}, \bar{\hat{v}}^{\nu}) \leq J_{\nu}(x, v_{\nu}, \bar{\hat{v}}^{\nu}), \forall \nu$$

for any admissible control v.

2.2 Existence of a Nash point

Our objective is to prove the following

Theorem 2.1 *We make the assumptions (2. 5), (2. 6),(2. 11), (2. 13),(2. 16). Suppose also*

(2. 19)
$$\theta \leq \frac{1}{2}, \neq -\frac{1}{N-1}, \ \text{or}\ \theta > 1$$

then there exists a Nash point for the game defined by the functionals (2. 14).

As indicated in the introduction, the method consists in considering a system of Bellman equations for the value functions of the game. This means here, that for a convenient control $\hat{v}$, possibly depending on x, the functions

(2. 20)
$$u_\nu(x) = J_\nu(x, \hat{v})$$

are the solutions of a system of partial differential equations. This system will permit to characterize optimal feedbacks for the N players. The proof of optimality will be performed by a verification argument. A key point is to obtain sufficient regularity properties for the value functions, otherwise it is not possible to obtain feedbacks. Techniques of partial differential equations are instrumental in obtaining these necessary regularity properties.

3 Bellman Equations

3.1 Notation

Introduce here the Lagrangians

(3. 21)
$$L_\nu(v,p) = \frac{1}{2}|v_\nu|^2 + \theta v_\nu . \bar{v}_\nu + p_\nu . \sum_\mu v_\mu$$

where

$$p = (p_1, \cdots, p_N).$$

The first point is to consider , for a given p, a Nash point in v for the functions $L_\nu(v,p)$. Clearly, the following conditions must hold (by differentiation) for such a Nash point $v(p)$

(3. 22)
$$v_\nu(p) + \theta \bar{v}_\nu(p) + p_\nu = 0.$$

Provided

(3. 23)
$$\theta \neq 1, \ \theta \neq -\frac{1}{N-1}$$

it is easy to check that the system (3. 22) has a unique solution given by the formulas

(3. 24)
$$v_\nu(p) = \frac{\theta \sum_\mu p_\mu}{(1-\theta)(1+(N-1)\theta)} - \frac{p_\nu}{1-\theta}.$$

We note also the complementary formulas

(3. 25)
$$\bar{v}_\nu(p) = \frac{-\sum_\mu p_\mu}{(1-\theta)(1+(N-1)\theta)} + \frac{p_\nu}{1-\theta}.$$

We can then define the quantities

(3. 26)
$$L_\nu(p) = L_\nu(v(p), p)$$

It is useful to also express, from (3. 24) and (3. 25), the vectors p_ν in terms of $v(p)$ as follows

(3. 27)
$$p_\nu = -v_\nu(p) - \theta \bar{v}_\nu(p)$$

and also

(3. 28) $$\bar{p}_\nu = -(N-1)\theta v_\nu(p) + (N\theta - 1)\bar{v}_\nu(p)$$

We can, in particular, write

(3. 29) $$L_\nu(p) = -\frac{1}{2}|v_\nu(p)|^2 - \frac{1}{\theta}p_\nu.(p_\nu + v_\nu(p))$$

The Bellman equations are written as follows

(3. 30) $$\begin{aligned} &-\frac{1}{2}\Delta u_\nu - g(x).Du_\nu + cu_\nu = f_\nu(x) + L_\nu(Du) \\ &u_\nu = 0, \text{ on } \partial\mathcal{O} \end{aligned}$$

3.2 Verification Property

We begin by stating the following result concerning the system of Bellman equations

Theorem 3.1 *We make the assumptions of Theorem 2.1, then there exists a solution of the system (3. 30), such that*

(3. 31) $$u_\nu \in W^{2,s}(\mathcal{O}),\ \forall 2 \leq s < \infty$$

In particular the functions u_ν are continuously differentiable, with second derivatives in $L^s(\mathcal{O})$.

We pospone the proof to the next section. From this regularity result we can derive the PROOF of THEOREM 2.1:
Consider

$$\hat{v}_\nu(x) = v_\nu(Du(x)).$$

It will correspond to an optimal feedback for the player ν. Next, we set

(3. 32) $$\hat{v}_\nu(t) = \hat{v}_\nu(x(t))$$

which defines a stochastic process, depending on x, the initial value of $x(t)$. Since, from the regularity of u, the functions $\hat{v}_\nu(x)$ are continuous, and recalling that $x(t)$ refers to the process, stopped at the exit of $\mathcal{O}$, we get that $\hat{v}_\nu(t)$ is a bounded adapted process. Define $\beta_{x,\hat{v}}(t)$, $P_{x,\hat{v}}$, $w_{x,\hat{v}}(t)$, by formulas (2. 7),(2. 8),(2. 9) then the system $\Omega, \mathcal{A}, \mathcal{F}^t, P_{x,\hat{v}}, w_{x,\hat{v}}(t)$ forms a probability system in which $w_{x,\hat{v}}(t)$ is an $\mathcal{F}^t$ standardized Wiener process. Note that from (2. 9) one has

(3. 33) $$dx = (g(x(t)) + \sum_\mu \hat{v}_\mu(t))\,dt + dw_{x,\hat{v}}(t),\quad x(0) = x.$$

Furthermore, from (3. 21), one has

$$L_\nu(Du(x)) = \frac{1}{2}|\hat{v}_\nu(x)|^2 + \theta\hat{v}_\nu(x).\bar{\hat{v}}_\nu(x) + Du_\nu(x).\sum_\mu \hat{v}_\mu(x)$$

hence, from (3. 30)

(3. 34) $$\begin{aligned} &-\frac{1}{2}\Delta u_\nu - g(x).Du_\nu + cu_\nu = f_\nu(x) + \\ &+\frac{1}{2}|\hat{v}_\nu(x)|^2 + \theta\hat{v}_\nu(x).\bar{\hat{v}}_\nu(x) + Du_\nu(x).\sum_\mu \hat{v}_\mu(x) \\ &u_\nu = 0, \text{ on } \partial\mathcal{O} \end{aligned}$$

From the regularity of u_ν, one may use Ito's formula to assert, taking account of (3. 33), (3. 34) and notation (3. 32)

$$d(u_\nu(x(t))e^{-ct}) = -e^{-ct}[f_\nu(x(t)) + \frac{1}{2}|\hat{v}_\nu(t)|^2 + \theta\hat{v}_\nu(t).\bar{\hat{v}}_\nu(t)]dt + e^{-ct}Du_\nu(x(t)).dw_{x,\hat{v}}(t)$$

and thus, integrating between 0 and τ_x, then taking the mathematical expectation, with respect to $P_{x,\hat{v}}$, we obtain easily

(3. 35) $$u_\nu(x) = J_\nu(x, \hat{v}) = J_\nu(x, \hat{v}_\nu, \bar{\hat{v}}^\nu)$$

Next, we notice that from the definition of Nash points

(3. 36) $$L_\nu(p) \le L_\nu(v_\nu, \bar{v}(p)^\nu), \forall v_\nu$$

hence also, from (3. 30) and (3. 34), we can state

(3. 37) $$\begin{aligned} &-\frac{1}{2}\Delta u_\nu - g(x).Du_\nu + cu_\nu \le f_\nu(x) + \\ &+\frac{1}{2}|v_\nu|^2 + \theta v_\nu.\bar{\hat{v}}_\nu(x) + Du_\nu(x).(v_\nu + \bar{\hat{v}}_\nu(x)), \forall v_\nu \\ &u_\nu = 0, \text{ on } \partial\mathcal{O} \end{aligned}$$

Consider then the control $(v_\nu(t), \bar{\hat{v}}^\nu(t))$, where $v_\nu(t)$ is any bounded, adaptive process, with values in R^n, and $\bar{\hat{v}}^\nu(t)$ is the part of $\hat{v}(t)$, without the component ν, as above. Then define $\beta_{x,v_\nu,\bar{\hat{v}}^\nu}(t)$, $P_{x,v_\nu,\bar{\hat{v}}^\nu}$, $w_{x,v_\nu,\bar{\hat{v}}^\nu}(t)$, by formulas (2. 7),(2. 8),(2. 9) then the system $\Omega, \mathcal{A}, \mathcal{F}^t, P_{x,v_\nu,\bar{\hat{v}}^\nu}, w_{x,v_\nu,\bar{\hat{v}}^\nu}(t)$ forms a probability system in which $w_{x,v_\nu,\bar{\hat{v}}^\nu}(t)$ is an $\mathcal{F}^t$ standardized Wiener process. Note again,that from (2. 9) one has

(3. 38) $$dx = (g(x(t)) + v_\nu(t) + \bar{\hat{v}}_\nu(t))\,dt + dw_{x,v_\nu,\bar{\hat{v}}^\nu}(t), \quad x(0) = x.$$

It follows from (3. 37) and Ito's formula that

$$d(u_\nu(x(t))e^{-ct}) \ge -e^{-ct}[f_\nu(x(t)) + \frac{1}{2}|v_\nu(t)|^2 + \theta v_\nu(t).\bar{\hat{v}}_\nu(t)]dt + e^{-ct}Du_\nu(x(t)).dw_{x,v_\nu,\bar{\hat{v}}^\nu}(t).$$

Thus, integrating between 0 and τ_x, then taking the mathematical expectation, with respect to $P_{x,v_\nu,\bar{\hat{v}}^\nu}$, we obtain easily

(3. 39) $$u_\nu(x) \le J_\nu(x, \hat{v}_\nu, \bar{\hat{v}}^\nu)$$

which implies that (2. 18) is satisfied, and thus the proof of Theorem 2.1 has been completed. ♠

4 Proof of Theorem 3.1

4.1 A priori estimates. The L^∞ bound

Suppose, we have a solution of (3. 30) which is in $(L^\infty(\mathcal{O} \cap H_0^1(\mathcal{O}))^N$, then we can check a priori estimates in that functional space. We begin with the L^∞ bound. It is a consequence of the maximum principle, which we check only formally (i.e. for a smooth solution) to shorten the argument. Recalling (3. 29) we have

$$-\frac{1}{2}\Delta u_\nu(x^*) + cu_\nu(x^*) \le f_\nu(x^*) \text{ if } Du_\nu(x^*) = 0$$

and thus if u_ν has a positive maximum (necessarily in the interior of $\mathcal{O}$), which we denote by x^*, we have

$$cu_\nu(x^*) \le f_\nu(x^*).$$

Therefore, clearly,

(4. 40) $$u_\nu(x) \le \frac{||f_\nu||_\infty}{c}$$

Consider the case $\theta \le \frac{1}{2}$. We define

$$\tilde{u} = \sum_\mu u_\mu.$$

We are going to sum up equations (3. 30), with respect to ν. We first express $L_\nu(p)$ in a different way. Using the formula (3. 24), in (3. 29), we obtain, after easy calculations

$$(4.\ 41)\qquad \begin{array}{l} L_\nu(p) = -\dfrac{\theta^2}{2(1-\theta)^2(1+(N-1)\theta)^2}|\sum_\mu p_\mu|^2 + \\ \dfrac{1-2\theta}{2(1-\theta)^2}|p_\nu|^2 + \dfrac{2\theta-1}{(1-\theta)^2(1+(N-1)\theta)}p_\nu.\sum_\mu p_\mu \end{array}$$

Hence, summing up in ν, we get

$$(4.\ 42)\qquad \begin{array}{l} \sum_\nu L_\nu(p) = \dfrac{\theta^2(3N-4)-2\theta(N-3)-2}{2(1-\theta)^2(1+(N-1)\theta)^2}|\sum_\mu p_\mu|^2 + \\ +\dfrac{1-2\theta}{2(1-\theta)^2}\sum_\nu |p_\nu|^2 \end{array}$$

hence, from the assumption on θ, it follows

$$\sum_\nu L_\nu(p) \geq \frac{\theta^2(3N-4)-2\theta(N-3)-2}{2(1-\theta)^2(1+(N-1)\theta)^2}|\sum_\mu p_\mu|^2.$$

Therefore, turning to the function $\tilde{u}$, we deduce

$$-\frac{1}{2}\Delta\tilde{u} - g(x).D\tilde{u} + c\tilde{u} \geq \sum_\nu f_\nu(x) +$$

$$+\frac{\theta^2(3N-4)-2\theta(N-3)-2}{2(1-\theta)^2(1+(N-1)\theta)^2}|D\tilde{u}|^2$$

and, of course,

$$\tilde{u} = 0, \text{ on } \partial\mathcal{O}.$$

Therefore, at a point of negative minimum of $\tilde{u}$ (necessarily inside $\mathcal{O}$), called x', one has

$$c\tilde{u}(x') \geq \sum_\nu f_\nu(x').$$

Hence

$$(4.\ 43)\qquad \sum_\mu u_\mu(x) \geq -\frac{||\sum_\mu f_\mu||_\infty}{c}$$

Combining (4. 40) and (4. 43), we deduce

$$(4.\ 44)\qquad -\frac{||f_\nu||_\infty}{c} - 2\sum_{\mu\neq\nu}\frac{||f_\mu||_\infty}{c} \leq u_\nu(x) \leq \frac{||f_\nu||_\infty}{c}$$

Assume now $\theta > 1$. Note that from (3. 25), we can also write

$$(4.\ 45)\qquad \bar{v}_\nu(p) = \frac{-\bar{p}_\nu + (N-1)\theta p_\nu}{(1-\theta)(1+(N-1)\theta)}.$$

and from (3. 29), (3. 27) it is easy to obtain

$$(4.\ 46)\qquad L_\nu(p) = -\frac{1}{2}|v_\nu(p) + \bar{v}_\nu(p)|^2 + (\frac{1}{2}-\theta)|\bar{v}_\nu(p)|^2$$

We define now

$$\tilde{u}_\nu = (N-1)\theta u_\nu - \sum_{\mu\neq\nu} u_\mu.$$

This combination is inspired from the form (4. 45). To derive a system for the functions $\tilde{u}_\nu$, we compute

$$(N-1)\theta L_\nu(p) - \sum_{\mu\neq\nu} L_\mu(p) = \frac{(N-1)\theta}{2}|v_\nu+\bar{v}_\nu|^2 + (\theta-\frac{1}{2})\sum_{\mu\neq\nu}|v_\mu|^2 +$$

$$-(\theta-\frac{1}{2})((N-1)\theta+2)|\bar{v}_\nu|^2 + (1-2\theta)v_\nu.\bar{v}_\nu.$$

This expression is obtained, after some algebraic manipulations, that we skip. Note that it implies

$$(N-1)\theta L_\nu(p) - \sum_{\mu\neq\nu} L_\mu(p) \geq -(\theta-\frac{1}{2})((N-1)\theta+2)|\bar{v}_\nu|^2 + (1-2\theta)v_\nu.\bar{v}_\nu. \tag{4. 47}$$

We can then derive inequalities for $\tilde{u}_\nu$, as follows

$$\begin{aligned} &-\frac{1}{2}\Delta\tilde{u}_\nu - g(x).D\tilde{u}_\nu + c\tilde{u}_\nu \geq (N-1)\theta f_\nu - \sum_{\mu\neq\nu} f_\mu + \\ &-\frac{(\theta-\frac{1}{2})((N-1)\theta+2)}{(1-\theta)^2(1+(N-1)\theta)^2}|D\tilde{u}_\nu|^2 + \frac{(1-2\theta)}{(1-\theta)(1+(N-1)\theta)}v_\nu(Du).D\tilde{u}_\nu \end{aligned} \tag{4. 48}$$

By maximum principle arguments, as before, we deduce

$$\tilde{u}_\nu(x) \geq -\frac{||(N-1)\theta f_\nu - \sum_{\mu\neq\nu} f_\mu||_\infty}{c}.$$

But we can express the functions u_ν in terms of the $\tilde{u}_\nu$. We see easily that

$$u_\nu = \frac{((N-1)\theta-(N-2))\tilde{u}_\nu + \sum_{\mu\neq\nu}\tilde{u}_\mu}{(N-1)(\theta-1)(1+(N-1)\theta)}.$$

Since $\theta > 1$, we obtain that u_ν is bounded below. In particular

$$u_\nu(x) \geq -\frac{\theta((N-1)\theta-(N-3))}{c(\theta-1)(1+(N-1)\theta)}\sum_\mu ||f_\mu||_\infty \tag{4. 49}$$

4.2 A priori estimates. The H_0^1 bound

We proceed with the estimate in $H_0^1(\mathcal{O})$, depending on the L^∞ bound. As it was the case, for the L^∞ bound, it will be very useful to combine the equations, in order to make apparent some special structures. We shall use the notation

$$\tilde{p}_\nu = p_\nu - p_N, \nu = 1,\cdots,N-1;\ \tilde{p}_N = p_N$$

and

$$\tilde{L}_\nu(p) = L_\nu(p) - L_N(p), \nu = 1,\cdots,N-1;\ \tilde{L}_N(p) = L_N(p).$$

It is easy to check that , for $\nu < N$, one has

$$\tilde{L}_\nu(p) = \frac{\frac{1}{2}-\theta}{1-\theta}(\bar{v}_\nu(p)+\bar{v}_N(p)).\tilde{p}_\nu$$

$$= \frac{\frac{1}{2}-\theta}{1-\theta}(\bar{v}_1(p)+\bar{v}_N(p)).\tilde{p}_\nu +$$

$$\frac{\frac{1}{2}-\theta}{(1-\theta)^2}(\tilde{p}_\nu-\tilde{p}_1).\tilde{p}_\nu.$$

The purpose of these manipulations is to consider the system of equations, related to the functions

$$\tilde{u}_\nu = u_\nu - u_N, \nu = 1, \cdots, N-1;\ \tilde{u}_N = u_N.$$

We arrive, at the following special structure of equations, omitting the ˜ symbol, to simplify the notation

$$\begin{aligned} &-\frac{1}{2}\Delta u_\nu - g(x).Du_\nu + cu_\nu = f_\nu + Q(Du).Du_\nu + H_\nu(Du) \\ &u_\nu = 0, \text{ on } \partial\mathcal{O} \end{aligned} \tag{4. 50}$$

where the functions $Q(p)$ and $H_\nu(p)$ satisfy the assumptions

$$\begin{aligned} &|Q(p)| \leq C|p| \\ &|H_\nu(p)| \leq K\sum_{\mu\leq\nu}|p_\mu|^2 \end{aligned} \tag{4. 51}$$

this is what we shall refer to as the special structure of the system. This special structure was not in the original formulation, but as we have seen, can be obtained by combining the equations. From the L^∞ bound, we can also assume that

$$|u_\nu(x)| \leq \rho \tag{4. 52}$$

To obtain a priori estimates, one uses a specific test function. Set

$$\beta(s) = e^s - s - 1$$

and

$$F = \Pi_{\nu=1}^N \exp\beta(\lambda_\nu u_\nu)$$

where λ_ν is a positive constant to be defined later. We have

$$DF = F\sum_{\nu=1}^N \lambda_\nu\beta'(\lambda_\nu u_\nu)Du_\nu.$$

We test (4. 50) with $F\lambda_\nu\beta'(\lambda_\nu u_\nu)$, inegrate by parts and add up. We get

$$\sum_\nu \frac{1}{2}\int_\mathcal{O}\lambda_\nu^2|Du_\nu|^2 e^{\lambda_\nu u_\nu}F\,dx + \frac{1}{2}\int_\mathcal{O}\frac{|DF|^2}{F}\,dx =$$

$$\int_\mathcal{O} Q.DF\,dx + \int_\mathcal{O}\sum_\nu \lambda_\nu(H_\nu(Du) - g.Du_\nu - cu_\nu - f_\nu)F(e^{\lambda_\nu u_\nu} - 1)\,dx.$$

Hence, also

$$\begin{aligned} &\sum_\nu \frac{1}{2}\int_\mathcal{O}\lambda_\nu^2|Du_\nu|^2 e^{\lambda_\nu u_\nu}F\,dx \leq \frac{1}{2}\int_\mathcal{O} FQ.Q\,dx + \\ &+\int_\mathcal{O}\sum_\nu \lambda_\nu(H_\nu(Du) - g.Du_\nu - cu_\nu - f_\nu)F(e^{\lambda_\nu u_\nu} - 1)\,dx. \end{aligned} \tag{4. 53}$$

The next step is to introduce the function

$$X = \Pi_{\nu=1}^N(\exp\beta(\lambda_\nu u_\nu) + \exp\beta(-\lambda_\nu u_\nu))$$

and the related quantities

$$X_\nu = X\frac{e^{\lambda_\nu u_\nu}\exp\beta(\lambda_\nu u_\nu) + e^{-\lambda_\nu u_\nu}\exp\beta(-\lambda_\nu u_\nu)}{\exp\beta(\lambda_\nu u_\nu) + \exp\beta(-\lambda_\nu u_\nu)}$$

$$\tilde{X}_\nu = X\frac{(e^{\lambda_\nu u_\nu} - 1)\exp\beta(\lambda_\nu u_\nu) - (e^{-\lambda_\nu u_\nu} - 1)\exp\beta(-\lambda_\nu u_\nu)}{\exp\beta(\lambda_\nu u_\nu) + \exp\beta(-\lambda_\nu u_\nu)}.$$

We have the inequalities

$$2^N \leq X \leq X_\nu \leq X e^{\lambda_\nu u_\nu}$$

$$|\tilde{X}_\nu| \leq X_\nu.$$

Applying relations similar to (4. 53), with λ_ν changed into $-\lambda_\nu$,one at a time, and summing up the 2^N relations obtained in this way, we get the inequality

$$\begin{aligned}(4.\ 54)\qquad &\sum_\nu \frac{1}{2}\int_{\mathcal{O}} \lambda_\nu^2 |Du_\nu|^2 X_\nu \, dx \leq \frac{1}{2}\int_{\mathcal{O}} XQ.Q \, dx + \\ &+\int_{\mathcal{O}} \sum_\nu \lambda_\nu (H_\nu(Du) - g.Du_\nu - cu_\nu - f_\nu)\tilde{X}_\nu \, dx.\end{aligned}$$

Using the assumptions (4. 51), recalling (4. 52) and the above inequalities, we can easily check the following inequality

$$\sum_\nu \int_{\mathcal{O}} |Du_\nu|^2 (\frac{1}{2}\lambda_\nu^2 - \frac{1}{2}\lambda_\nu - K\sum_{\mu>\nu} \lambda_\mu e^{\rho\lambda_\mu} - K\lambda_\nu - \frac{1}{2}C^2)\, dx \leq K_0(\rho).$$

Therefore, if we pick the constants $\frac{1}{2}$, so that

$$\frac{1}{2}\lambda_\nu^2 - \frac{1}{2}\lambda_\nu - K\sum_{\mu>\nu} \lambda_\mu e^{\rho\lambda_\mu} - K\lambda_\nu - \frac{1}{2}C^2 > 1, \forall \nu$$

which is possible, defining the constants backwards (starting with u_N, we obtain

$$\int_{\mathcal{O}} |Du|^2 \, dx \leq K_0(\rho).$$

Thus the H_0^1 estimate has been obtained.

4.3 Completion of Proof

The starting point is to consider an approximation, u_ν^ϵ of the functions u_ν, defined by the system of equations

$$\begin{aligned}(4.\ 55)\qquad &-\frac{1}{2}\Delta u_\nu^\epsilon - g(x).Du_\nu^\epsilon + cu_\nu^\epsilon = f_\nu(x) + \frac{L_\nu(Du)}{1+\epsilon|L(Du)|} \\ &u_\nu = 0, \text{ on } \partial\mathcal{O}\end{aligned}$$

The right hand side of (4. 55) being bounded, it is classical to prove the existence of a solution of (4. 55), which belongs to $W^{2,s}(\mathcal{O})$, $\forall 2 \leq s < \infty$. Writing

$$L_\nu^\epsilon(p) = \frac{L_\nu(p)}{1+\epsilon|L(p)|}$$

it is easy to check that the method to obtain bounds, described in the two previous sections, is applicable to (4. 55). In particular, we can perform the manipulation of equations, and obtain, in this way, L^∞ and H_0^1 bounds, which are uniform in ϵ. The main step is to obtain C^δ estimates. This is obtained, by using specific test functions, similar, although more involved, than those already used for the H_0^1 bounds. We refer to the book of the authors, see [2]. With such estimates, one can let ϵ tend to 0, and obtain a solution of (3. 30). Once a solution in $C^\delta \cap H_0^1$ is established, it is possible, by classical arguments to show the $W^{2,s}$ regularity, see Ladyzenskaya-Ural'tseva [4].

5 The case of two players with different coupling terms in the payoffs

When $N=2$ the Lagrangians (3. 21) take the form

$$
\begin{aligned}
L_1(v,p) &= \frac{1}{2}|v_1|^2 + \theta v_1.v_2 + p_1.(v_1+v_2) \\
L_2(v,p) &= \frac{1}{2}|v_2|^2 + \theta v_1.v_2 + p_2.(v_1+v_2)
\end{aligned}
\tag{5. 56}
$$

and Theorem 2.1 amounts to: if

$$
\theta \leq \frac{1}{2}, \neq -1, \text{ or } \theta > 1 \tag{5. 57}
$$

then there exists a Nash point for the game defined by the functionals (2. 14). If we examin carefully the proof, see section 4, then we made the following combinations of equations. In the case $\theta \leq \frac{1}{2}, \neq -1$, then we simply added up the equations to derive the L^∞ bound from below. When $\theta > 1$, we computed

$$\theta L_1(p) - L_2(p)$$

$$\theta L_2(p) - L_1(p)$$

to derive bounds from below for

$$\tilde{u}_1 = \theta u_1 - u_2$$

$$\tilde{u}_2 = \theta u_2 - u_1$$

and since

$$u_1 = \frac{\theta \tilde{u}_1 + \tilde{u}_2}{\theta^2 - 1}$$

$$u_2 = \frac{\theta \tilde{u}_2 + \tilde{u}_1}{\theta^2 - 1}$$

the same type of bound from below holds for u_1, u_2. To realize the special structure (4. 50), (4. 51) we just substract the equations and consider the new variables

$$\tilde{u}_1 = u_1 - u_2, \tilde{u}_2 = u_2.$$

In this section, we shall consider different coupling terms in the two Lagrangians, and see how our existence results extend.

5.1 Description of the model and statement of results

We consider the following payoffs, for two players,

$$
\begin{aligned}
J_1(x,v) &= E_{x,v}\int_0^{\tau_x} e^{-ct}(f_1(x(t)) + \frac{1}{2}|v_1(t)|^2 + \theta v_1(t).v_2(t))dt \\
J_2(x,v) &= E_{x,v}\int_0^{\tau_x} e^{-ct}(f_2(x(t)) + \frac{1}{2}|v_2(t)|^2 + \sigma v_1(t).v_2(t))dt
\end{aligned}
\tag{5. 58}
$$

This model leads to the following Lagrangians

$$
\begin{aligned}
L_1(v,p) &= \frac{1}{2}|v_1|^2 + \theta v_1.v_2 + p_1.(v_1+v_2) \\
L_2(v,p) &= \frac{1}{2}|v_2|^2 + \sigma v_1.v_2 + p_2.(v_1+v_2)
\end{aligned}
\tag{5. 59}
$$

We shall need the condition

$$
\sigma\theta \neq 1 \tag{5. 60}
$$

We then state the following conditions

(5. 61)
$$\sigma > 0, \theta > 0, \ \ \sigma\theta > 1$$

or

(5. 62)
$$\begin{gathered} 0 < \theta \le \frac{1}{2}, \text{ and} \\ \frac{1-\theta-(1+\theta)\sqrt{1-2\theta}}{2\theta} \le \sigma \le -(\theta^2+\theta+1)+(\theta+1)\sqrt{\theta^2+2} \end{gathered}$$

or

(5. 63)
$$\begin{gathered} -1 < \theta < 0, \text{ and} \\ \frac{1-\theta-(1+\theta)\sqrt{1-2\theta}}{2\theta} \le \sigma \le -(\theta^2+\theta+1)+(\theta+1)\sqrt{\theta^2+2} \end{gathered}$$

or

(5. 64)
$$\begin{gathered} \theta < -1, \text{ and} \\ -(\theta^2+\theta+1)+(\theta+1)\sqrt{\theta^2+2} \le \sigma \le \frac{1-\theta-(1+\theta)\sqrt{1-2\theta}}{2\theta} \end{gathered}$$

The case $\theta = 0$ can be obtained as a limit case in (5. 63), which yields

(5. 65)
$$\theta = 0, -\frac{1}{2} \le \sigma \le -1+\sqrt{2}$$

The case $\theta = -1$ can be obtained as a limit case in (5. 63) and (5. 64), which yields

(5. 66)
$$\theta = -1, \sigma = -1$$

but this is forbidden by the condition (5. 60). We then state the following

Theorem 5.1 *We make the assumptions (2. 5), (2. 6),(2. 11), (2. 13),(2. 16), with $N = 2$. Suppose also that the parameters θ, σ satisfy (5. 60) and one of the conditions (5. 61), (5. 62), (5. 63), (5. 64), (5. 65) then there exists a Nash point for the game defined by the functionals (5. 58).*

Note that except in the case (5. 61), we have $\sigma \le \frac{1}{2}$. This can be checked by verifying that in all cases (5. 62),(5. 63), (5. 64), (5. 65) the upper bound on σ is always smaller or equal to $\frac{1}{2}$. We can also check, that the case $\sigma = \theta$ reduces to results already obtained.
We shall not give a full proof of Theorem 5.1, since many of the steps are in common with that of Theorem 2.1. The main differences concern the way we arrive at the L^∞ bounds , and at the special structure which leads to the H^1_0 and C^δ bounds. These two aspects are dealt with in the next two sections, respectively.

5.2 L^∞ bounds

We present here the equivalent of section 4.1. First, we notice that a Nash point for the Lagrangians $L_1(v,p), L_2(v,p)$, see (5. 59) yields

(5. 67)
$$\begin{gathered} v_1(p) + \theta v_2(p) + p_1 = 0. \\ v_2(p) + \sigma v_1(p) + p_2 = 0. \end{gathered}$$

hence

(5. 68)
$$\begin{aligned} L_1(p) = L_1(v(p),p) &= -\frac{1}{2}|v_1(p)|^2 + p_1.v_2(p) \\ L_2(p) = L_2(v(p),p) &= -\frac{1}{2}|v_2(p)|^2 + p_2.v_1(p). \end{aligned}$$

From this form, we get the equivalent of (4. 40). So we only need bounds from below. Note that, provided $\sigma\theta \neq 1$, we can solve the system (5. 67), and obtain

$$\begin{aligned} v_1(p) &= \frac{-p_1+\theta p_2}{1-\sigma\theta} \\ v_2(p) &= \frac{-p_2+\sigma p_1}{1-\sigma\theta}. \end{aligned} \tag{5. 69}$$

We now consider the case (5. 61) and set

$$\begin{aligned} \tilde{u}_1 &= \sigma u_1 - u_2 \\ \tilde{u}_2 &= \theta u_2 - u_1. \end{aligned} \tag{5. 70}$$

from which, we deduce

$$\begin{aligned} u_1 &= \frac{\theta\tilde{u}_1+\tilde{u}_2}{\sigma\theta-1} \\ u_2 &= \frac{\sigma\tilde{u}_2+\tilde{u}_1}{\sigma\theta-1}. \end{aligned} \tag{5. 71}$$

Thanks to conditions (5. 61), it is thus sufficient to obtain bounds from below on $\tilde{u}_1, \tilde{u}_2$, to derive similar ones on u_1, u_2. After easy computations, one can derive the following expressions

$$\begin{aligned} \sigma L_1(p) - L_2(p) &= \frac{\sigma}{2}|v_1(p)+v_2(p)|^2 + |v_2(p)|^2\left(-\frac{\sigma}{2}-\sigma\theta+\frac{1}{2}\right) + (1-2\sigma)v_1(p).v_2(p) \\ \theta L_2(p) - L_1(p) &= \frac{\theta}{2}|v_1(p)+v_2(p)|^2 + |v_1(p)|^2\left(-\frac{\theta}{2}-\sigma\theta+\frac{1}{2}\right) + (1-2\theta)v_1(p).v_2(p). \end{aligned} \tag{5. 72}$$

Combining Bellman equations, we arrive at

$$\begin{aligned} &-\frac{1}{2}\Delta\tilde{u}_1 - g(x).D\tilde{u}_1 + c\tilde{u}_1 \geq \sigma f_1 - f_2 + \\ &+|D\tilde{u}_1|^2\frac{-\frac{\sigma}{2}-\sigma\theta+\frac{1}{2}}{(1-\sigma\theta)^2} + \frac{1-2\sigma}{(1-\sigma\theta)^2}D\tilde{u}_1.D\tilde{u}_2 \\ &-\frac{1}{2}\Delta\tilde{u}_2 - g(x).D\tilde{u}_2 + c\tilde{u}_2 \geq \theta f_2 - f_1 + \\ &+|D\tilde{u}_2|^2\frac{-\frac{\sigma}{2}-\sigma\theta+\frac{1}{2}}{(1-\sigma\theta)^2} + \frac{1-2\sigma}{(1-\sigma\theta)^2}D\tilde{u}_1.D\tilde{u}_2 \end{aligned} \tag{5. 73}$$

from which bounds from below on $\tilde{u}_1, \tilde{u}_2$ are easily obtained.
We want now to consider the sum

$$\tilde{u} = u_1 + u_2 \tag{5. 74}$$

which implies computing the sum

$$L_1(p) + L_2(p).$$

After easy computations, whose details are left to the reader, we can check the following expression

$$\begin{aligned} L_1(p)+L_2(p) &= -\frac{1}{2}\frac{\theta^2+\sigma^2}{(1-\sigma\theta)^2}|p_1+p_2|^2 + \frac{1}{(1-\sigma\theta)^2}(p_1+p_2)((\sigma+\theta+1)(\theta p_1+\sigma p_2)-(p_1+p_2)) \\ &+\frac{1}{2(1-\sigma\theta)^2}[p_1^2(2(\sigma+1)(1-\sigma\theta)-(\theta+1)^2)+p_2^2(2(\theta+1)(1-\sigma\theta)-(\sigma+1)^2)] \end{aligned} \tag{5. 75}$$

In order to realize

$$L_1(p)+L_2(p) \geq 0, \text{ when } p_1+p_2=0$$

we must assume the inequalities

$$2(\sigma+1)(1-\sigma\theta)-(\theta+1)^2 \geq 0$$

$$2(\theta+1)(1-\sigma\theta)-(\sigma+1)^2 \geq 0$$

which rewritten as second order polynomials in σ yields

$$\begin{aligned} 2\sigma^2\theta+2\sigma(\theta-1)+\theta^2+2\theta-1 \leq 0 \\ \sigma^2+2\sigma(\theta^2+\theta+1)-(2\theta+1) \leq 0 \end{aligned} \tag{5. 76}$$

Note that, if $\theta > 0$, the first condition implies that there must exist roots for the 2nd order polynomial in σ, and writting the discriminant, we obtain easily

$$\theta \leq \frac{1}{2}.$$

and similarly

$$\sigma \leq \frac{1}{2}.$$

So we necessarily limit ourselves to the possibilities concerning θ described in (5. 62), (5. 63), (5. 64), (5. 65). Note that the second polynomial has always roots in σ. We first consider the case (5. 65). In this case,$\theta = 0$, the conditions (5. 76) reduce to (5. 65). So, there remains the three possibilities expressed in (5. 62), (5. 63), (5. 64). Note that, since the second polynomial has roots in σ, we derive the conditions

$$-(\theta^2+\theta+1)-|\theta+1|\sqrt{\theta^2+2} \leq \sigma \leq -(\theta^2+\theta+1)+|\theta+1|\sqrt{\theta^2+2}. \tag{5. 77}$$

The roots of the first polynomial are

$$\frac{-(\theta-1)\pm|\theta+1|\sqrt{1-2\theta}}{2\theta}$$

but, to state the conditions on σ, we have to distinguish the cases $\theta > 0$ from $\theta < 0$. So in the case (5. 62), we must write

$$\frac{1-\theta-(\theta+1)\sqrt{1-2\theta}}{2\theta} \leq \sigma \leq \frac{1-\theta+(\theta+1)\sqrt{1-2\theta}}{2\theta} \tag{5. 78}$$

and (5. 77) reduce to

$$-(\theta^2+\theta+1)-(\theta+1)\sqrt{\theta^2+2} \leq \sigma \leq -(\theta^2+\theta+1)+(\theta+1)\sqrt{\theta^2+2}. \tag{5. 79}$$

To proceed we shall need to study the sign of the following functions

$$\chi(\theta) = 2\theta(\sqrt{\theta^2+2}-\theta)-1-\sqrt{1-2\theta}$$

$$\phi(\theta) = 2\theta(\sqrt{\theta^2+2}-\theta)-1+\sqrt{1-2\theta}$$

$$\psi(\theta) = 2\theta(\sqrt{\theta^2+2}+\theta)+1+\sqrt{1-2\theta}.$$

A simple calculation shows that

$$\chi'(\theta) = 2\frac{(\sqrt{\theta^2+2}-\theta)^2}{\sqrt{\theta^2+2}}+(1-2\theta)^{\frac{-1}{2}} > 0$$

hence $\chi'(\theta) > 0$, and , since $\chi(\frac{1}{2}) = 0$, we can assert that

$$\chi(\theta) \leq 0, \quad \text{for } \theta \leq \frac{1}{2}. \tag{5. 80}$$

Next

$$\phi'(\theta) = 2\frac{(\sqrt{\theta^2+2}-\theta)^2}{\sqrt{\theta^2+2}} - (1-2\theta)^{\frac{-1}{2}} > 0$$

and

$$\phi"(\theta) = -2\frac{(\sqrt{\theta^2+2}-\theta)^2}{\sqrt{\theta^2+2}}[2+\frac{\theta}{\sqrt{\theta^2}}] - (1-2\theta)^{\frac{-3}{2}} > 0.$$

Therefore $\phi'(\theta)$ is decreasing, and since

$$\phi'(-\infty) = +\infty, \ \ \phi'(0) = 2\sqrt{2}-1, \ \ \phi'(\frac{1}{2}) = -\infty$$

it follows that the equation $\phi'(\theta) = 0$ has only one root θ_0, with

$$0 < \theta_0 < \frac{1}{2}.$$

Noting that

$$\phi(0) = \phi(\frac{1}{2}) = 0$$

we can assert that

(5. 81)
$$\begin{aligned} &\phi(\theta) \leq 0, \ \ \text{for } \theta \leq 0 \\ &\phi(\theta) \geq 0, \ \ \text{for } 0 \leq \theta \leq \frac{1}{2}. \end{aligned}$$

Finally, we have

(5. 82)
$$\psi(\theta) \geq 0, \ \ \text{for } \theta \leq \frac{1}{2}.$$

Indeed, the positivity being obvious when $\theta \geq 0$, it is sufficient to consider the case $\theta \leq 0$. But then writing

$$\psi(\theta) = -2\theta^2\sqrt{1+\frac{2}{\theta^2}} + 2\theta^2 + 1 + \sqrt{1-2\theta}$$

and using

$$\sqrt{1+\frac{2}{\theta^2}} \leq 1 + \frac{1}{\theta^2}$$

we check easily that

$$\psi(\theta) \geq -1 + \sqrt{1-2\theta} \geq 0.$$

With the properties (5. 80), (5. 81), (5. 82) in mind, we first analyze the case $0 \leq \theta \leq \frac{1}{2}$, in which (5. 78), (5. 79) apply. We check easily that

$$-(\theta^2+\theta+1) - (\theta+1)\sqrt{\theta^2+2} \leq \frac{1-\theta-(\theta+1)\sqrt{1-2\theta}}{2\theta}$$

and

$$-(\theta^2+\theta+1) + (\theta+1)\sqrt{\theta^2+2} \leq \frac{1-\theta+(\theta+1)\sqrt{1-2\theta}}{2\theta}$$

from which (5. 62) follows, checking also that the interval in which σ lies is not empty.
Turning now to the situation when $\theta < 0$, we first notice that the conditions on σ derived from the first polynomial are expressed as

$$\sigma \geq \frac{1-\theta-|\theta+1|\sqrt{1-2\theta}}{2\theta}$$

or

$$\sigma \leq \frac{1-\theta+|\theta+1|\sqrt{1-2\theta}}{2\theta}.$$

So we have to distinguish two cases. First

$$-1 < \theta < 0$$

on which we have the conditions

$$-(\theta^2+\theta+1)-(\theta+1)\sqrt{\theta^2+2} \le \sigma \le -(\theta^2+\theta+1)+(\theta+1)\sqrt{\theta^2+2}$$

and

$$\sigma \ge \frac{1-\theta-(\theta+1)\sqrt{1-2\theta}}{2\theta}$$

or

$$\sigma \le \frac{1-\theta+(\theta+1)\sqrt{1-2\theta}}{2\theta}.$$

Using the sign of the functions ϕ and ψ, one can check that

$$-(\theta^2+\theta+1)-(\theta+1)\sqrt{\theta^2+2} \le \frac{1-\theta-(\theta+1)\sqrt{1-2\theta}}{2\theta}$$

$$\le -(\theta^2+\theta+1)+(\theta+1)\sqrt{\theta^2+2}$$

and

$$\frac{1-\theta+(\theta+1)\sqrt{1-2\theta}}{2\theta} \le -(\theta^2+\theta+1)-(\theta+1)\sqrt{\theta^2+2}.$$

Comparing intervals, we conclude easily that (5. 63) holds. The last case is

$$\theta < -1$$

on which we have the conditions

$$-(\theta^2+\theta+1)+(\theta+1)\sqrt{\theta^2+2} \le \sigma \le -(\theta^2+\theta+1)-(\theta+1)\sqrt{\theta^2+2}$$

and

$$\sigma \ge \frac{1-\theta+(\theta+1)\sqrt{1-2\theta}}{2\theta}$$

or

$$\sigma \le \frac{1-\theta-(\theta+1)\sqrt{1-2\theta}}{2\theta}.$$

We proceed in a way similar to (5. 63) to obtain (5. 64).

5.3 H_0^1 bound

As in section 4.2, we shall derive the H_0^1 bound once the L^∞ bound is available. Similar methods,although more involved permit to derive the C^δ estimates, and will not be detailed here. Recalling the method of section 4.2, (so we consider here temporarily the case $\sigma = \theta$), for $N = 2$, we set

$$\tilde{u}_1 = u_1 - u_2$$

$$\tilde{u}_2 = u_2$$

$$\tilde{p}_1 = p_1 - p_2$$

$$\tilde{p}_2 = p_2$$

and we get

$$\tilde{L}_1(p) = L_1(p) - L_2(p)$$

$$= \frac{\frac{1}{2} - \theta}{1 - \theta}(v_1(p) + v_2(p)).\tilde{p}_1.$$

We note that $v_1(p) + v_2(p)$ is a linear function of p, hence of $\tilde{p}$, so we may set

$$Q(\tilde{p}) = \frac{\frac{1}{2} - \theta}{1 - \theta}(v_1(p) + v_2(p))$$

with

$$|Q(\tilde{p})| \leq C|\tilde{p}|.$$

Hence we get the structure

$$\tilde{L}_1(p) = Q(\tilde{p}).\tilde{p}_1.$$

We may write

$$\tilde{L}_2(p) = Q(\tilde{p}).\tilde{p}_2 + H_2(\tilde{p}),$$

with

$$|H_2(\tilde{p})| \leq K(|\tilde{p}_1|^2 + |\tilde{p}_2|^2).$$

With this special structure, we get the desired bounds on $\tilde{u}$.
Turning to our more general situation, where $\sigma \neq \theta$, we infer from the preceding argument, that if we can find a real number λ such that the following property holds

(5. 83) $$L_1(p) + \lambda L_2(p) = R(p).(p_1 + \lambda p_2)$$

with $R(p)$ a linear function, then the special struture detailed above is obtained, setting

$$\tilde{u}_1 = u_1 + \lambda u_2$$

$$\tilde{u}_2 = u_2$$

and related relations as above. When $\sigma = \theta$, the convenient value of λ is -1. So eveything amounts to proving that there exists λ, such that (5. 83) holds.
Using the definition of $L_1(p), L_2(p)$, see (5. 59), we can write

(5. 84) $$L_1(p) + \lambda L_2(p) = \frac{1}{2}(v_1^2 + \lambda v_2^2) + (\theta + \lambda\sigma)v_1.v_2 + (p_1 + \lambda p_2)(v_1 + v_2)$$

where we have omitted to write explicitly $v_1(p), v_2(p)$, which are linear functions of p. From (5. 67), we can also write

(5. 85) $$-(p_1 + \lambda p_2) = (1 + \lambda\sigma)v_1 + (\lambda + \theta)v_2$$

Considering the forms (5. 84),(5. 85), we see that the finding of λ amounts to finding three numbers λ, β, γ , such that the following identity holds

(5. 86) $$\frac{1}{2}(v_1^2 + \lambda v_2^2) + (\theta + \lambda\sigma)v_1.v_2 = ((1 + \lambda\sigma)v_1 + (\lambda + \theta)v_2)(\beta v_1 + \gamma v_2)$$

where in (eq:5.30) v_1, v_2 can take any values. Identifying terms on both sides of (5. 86) yields

$$\frac{1}{2} = \beta(1 + \lambda\sigma)$$

$$\frac{\lambda}{2} = (\lambda + \theta)\gamma$$

$$\theta + \lambda\sigma = (1 + \lambda\sigma)\gamma + (\lambda + \theta)\beta.$$

So we get obviously

$$\beta = \frac{1}{2(1+\lambda\sigma)}, \gamma = \frac{\lambda}{2(\lambda+\theta)}.$$

So the last relation yields

$$\theta + \lambda\sigma = \frac{(1+\lambda\sigma)\lambda}{2(\lambda+\theta)} + \frac{\lambda+\theta}{2(1+\lambda\sigma)}$$

which is an equation in λ. Setting

$$T(\lambda) = 2(\lambda+\theta)(1+\lambda\sigma)(\theta+\lambda\sigma) - \lambda(1+\lambda\sigma)^2 - (\lambda+\theta)^2$$

then the problem amounts to finding a real root of

$$T(\lambda) = 0.$$

We express $T(\lambda)$ as follows

$$T(\lambda) = \lambda^3\sigma^2 + \lambda^2(2\sigma\theta + 2\sigma^2\theta - 1) + \lambda(2\sigma\theta + 2\sigma\theta^2 - 1) + \theta^2.$$

It is obvious that this polynomial has at least one root. Note that, whenever $\sigma = \theta$, one has

$$T(\lambda) = \lambda^3\theta^2 + (\lambda^2 + \lambda)(2\theta^2 + 2\theta^3 - 1) + \theta^2$$

and we obtain that $\lambda = -1$ is a root.

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