Linear Random Functionals and Stochastic Calculus

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1 Introduction

Linear Random Functionals have been used extensively by the author [1], [2] to develop the theory of Kalman filtering for infinite dimensional linear systems. It is reminiscent of the concept of stochastic integral, which it partly generalizes. Stochastic calculus is developped in relation with Linear Random Functionals. We compare this approach to that of cylindrical Wiener processes, introduced by G. DA PRATO - J. ZACZYK [6]. We also consider some nonlinear problems, which seem to appear for the first time.

2 Linear Random Functionals

Let Φ be a separable Hilbert space and Φ' its dual.
Let $(\Omega, \mathcal{A}, P)$ be a probability space.
A Linear Random Functional (L.R.F.) on Φ' is a family $\zeta_{\varphi_*}(\omega)$ of real random variables, such that

(2.1) $$\varphi_* \to \zeta_{\varphi_*}(\omega) \quad \text{is a.s. linear} .$$

As a particular case we have

(2.2) $$\zeta_{\varphi_*}(\omega) = < \varphi_*,\ \zeta(\omega) >$$

where $\zeta(\omega)$ is a random variable with values in Φ. Note that the existence of $\zeta(\omega)$ such that (2.2) occurs is equivalent to the fact that the L.R.F. is a.s. continuous. When this property is not satisfied, it may happen, which will be generally assumed, that

(2.3) $$\varphi_* \to \zeta_{\varphi_*}(.) \in \mathcal{L}(\Phi'; L^2(\Omega, \mathcal{A}, P)) .$$

In that case, the quantities $E\zeta_{\varphi_*}$ and $E\zeta_{\varphi_*}\zeta_{\tilde{\varphi}_*}$ have a meaning, and moreover we can write

(2.4) $$E\zeta_{\varphi_*} = < m, \varphi_* > ,\ m \in \phi$$

(2.5) $$E\zeta_{\varphi_*}\zeta_{\tilde{\varphi}_*} - E\zeta_{\varphi_*}\ E\zeta_{\tilde{\varphi}_*} \ = \ < \Gamma\varphi_*, \tilde{\varphi}_* >$$

with $\Gamma \in \mathcal{L}(\Phi', \Phi)$, self-adjoint and positive.
We call m the mathematical expectation and Γ the covariance operator of the L.R.F. ζ_{φ_*}.
Let Ψ be another separable Hilbert space, and u be an affine map from Φ to Ψ, defined by

(2.6) $$u(\varphi) = \bar{\psi} + B\varphi, \quad \bar{\phi} \in \Psi, \quad B \in \mathcal{L}(\Phi, \Psi) .$$

Given ζ_{φ_*} a L.R.F. on Φ', we define the image $(u\zeta)_{\psi_*}$ as a L.R.F. on Ψ', by the formula

(2.7) $$(u\zeta)_{\psi_*}(\omega) = < \bar{\psi}, \psi_* > \ + \zeta_{B^*\psi_*}(\omega) .$$

It is easy to check that the mathematical expectation and the covariance of the image are given by the formulas

(2.8) $$E(u\zeta)_{\psi_*} \ = \ < \bar{\phi} + Bm,\ \psi_* >$$

(2.9) $$E(u\zeta)_{\psi_*}(u\zeta)_{\tilde\psi_*} - E(u\zeta)_{\psi_*}E(u\zeta)_{\tilde\psi_*} = < B\Gamma B^*\psi_*, \tilde\psi_* >$$

As a particular important case, we shall consider

$$\Phi = L^2(0,T;E) \ , \quad E \text{ separable Hilbert space}$$

and a L.R.F. on $\Phi' = L^2(0,T;E')$ denoted by $\xi_{e_*(.)}(\omega)$, where $e_*(.)$ is any element of $L^2(0,T;E')$, with the covariance operator

(2.10) $$< \Gamma e_*(.), \tilde e_*(.) > = \int_0^T < Q(t)e_*(t), \tilde e_*(t) > dt$$

where $Q(.) \in L^\infty(0,T;\mathcal{L}(E';E))$ self adjoint, positive.
Consider the example $E = E' = R^n$, $Q(t) = I$, and $\xi_{e(.)}(\omega)$ is gaussian, $\forall e(.) \in L^2(0,T;R^n)$, with mathematical expectation 0. Let $e_1, \cdots, e_n$ be an orthonormal basis of R^n, consider the stochastic process

$$\xi_{e_i 1_{(0,t)}(.)}(\omega) \ = \ w_i(t)$$

then

$$Ew_i(t)w_j(s) \ = \ \delta_{ij}\min(t,s)$$

and thus $w(t) = (w_1(t), \cdots, w_n(t))$ is a Wiener process, with values in R^n. It is easy to verify that

(2.11) $$\xi_{e(.)}(\omega) = \int_0^T e(t).dw(t)$$

where on the right hand side, we have the usual stochastic integral. This identification does not carry over to general Hilbert spaces, but a generalization of the stochastic integral is possible, as shown in the next section.
However when $Q(t)$ is nuclear, namely

$$\sum_i \int_0^T \ < Q(t)e_{*,i} \ , \ e_{*,i} > \ dt \ < \infty$$

for any orthonormal basis $e_{*,i}$ of E', then we can write

$$\xi_{e_*(.)}(\omega) \ = \ \int_0^T < e_*(t), dw(t) >$$

where

(2.12) $$w(t) = \sum_i w_i(t) J^{-1} e_{*,i}$$

with J isomorphism from E to E', and

$$w_i(t) = \xi_{e_{*,i} 1_{(0,t)}} \ .$$

The convergence (2.12) is in $L^2(\Omega, \mathcal{A}, P)$.
This analysis shows that the concept of L.R.F. generalizes the usual approach of the white noise based on the Wiener process and stochastic integral with deterministic integrand.
We shall show that the stochastic integral with adapted stochastic integrand can be generalized within the framework of L.R.F.s.

To cope with difficulty of the convergence of (2.12), G. DA PRATO, J. ZABCYK [6] have introduced the concept of cylindrical Wiener process. It follows from the following consideration. Call first, to simplify the notation

$$e_i = J^{-1} e_{*,i}$$

which is an orthonormal basis of E. Now, pick any sequence $\alpha_i \geq 0$, such that $\Sigma \alpha_i = 1$.

Define next

$$E_1 = \left\{ e = \sum \lambda_i e_i \mid \sum \lambda_i^2 \alpha_i < \infty \right\} \tag{2.13}$$

which we equip with the scalar product norm

$$\|e\|_1^2 \ = \ \sum_i \lambda_i^2 \alpha_i \ . \tag{2.14}$$

Obviously $E \subset E_1$, and the injection of E into E_1 is Hilbert Schmidt, since $\|e_i\|_1^2 = \alpha_i$. It is easy to check that

$$w(t) = \Sigma w_i(t) e_i \ \in \ L^\infty\Big(0, T; L^2(\Omega, \mathcal{A}, P; E_1)\Big) \tag{2.15}$$

and

$$E\|w(t)\|_1^2 = t \sum_i \alpha_i \ . \tag{2.16}$$

Then, we may take, as a definition

$$\xi_{e_*(t)}(\omega) \ = \ \sum_i \int_0^T < e_*(t), e_i > dw_i(t) \tag{2.17}$$

and indeed the formal sum on the right hand side of (2.17) converges in $L^2(\Omega, \mathcal{A}, P)$. Note that the space E_1 does not play any specific role in (2.17),

and that $E_1' \subset E'$, which implies that $< e_*(t), dw(t) >$ cannot be interpreted in the sense of the dualité E_1', E_1.
These considerations show that the right object to consider is the L.R.F. $\xi_{e_*(.)}(\omega)$.

3 Generalized stochastic integral

Consider a L.R.F. on $L^2(0,T;E')$, $\xi_{e_*(.)}(\omega)$, which is gaussian, with mathematical expectation 0, and covariance operator defined by (2.10). We consider a filtration $\mathcal{E}^t$ and we assume that

$$\text{(3.1)}\qquad \begin{aligned} &\xi_{e_*\mathbb{1}_{(0,s)}} \quad \text{is } \mathcal{E}^t \text{ measurable}, \forall e_* \in E',\ \forall s \le t \\ &\xi_{e_*\mathbb{1}_{(s,\theta)}} \quad \text{is independant of } \mathcal{E}^s\ \forall e_* \in E',\ \forall s \le t \end{aligned}$$

When

$$\mathcal{E}^t = \sigma\Big(\xi_{e_*\mathbb{1}_{(0,s)}},\ \forall e_* \in E',\ \forall s \le t\Big)$$

then the second property (3.1) follows from the gaussian assumption and the non correlation of $\xi_{e_*\mathbb{1}_{(s,\theta)}}$ with the variables generating $\mathcal{E}^s$.
As in the usual case of stochastic integrals, we shall extend the definition of $\xi_{e_*(.)}(\omega)$ to $\xi_{e_*(.;\omega)}(\omega)$, where $e_*(t;\omega)$ is a stochastic process with values in E', adapted to the filtration $\mathcal{E}^t$, such that

$$\text{(3.2)}\qquad E\int_0^T \|e_*(t)\|^2\, dt \ <\ \infty$$

We shall prove the following

Theorem 3.1 *Consider $\xi_{e_*(.)}(\omega)$ and $e_*(.;\omega)$ as above and assume (3.1). Then $\xi_{e_*(.;\omega)}(\omega)$ is defined as a random variable in $L^2(\Omega,\mathcal{A},P)$, and one has*

$$\text{(3.3)}\qquad E\ \xi_{e_*(.;.)}(.) = 0$$

$$\text{(3.4)}\qquad E\Big(\xi_{e_*(.;.)}(.)\Big)^2 = E\int_0^T < Q(\tau)e_*(\tau;\omega), e_*(\tau;\omega) > d\tau$$

Proof.

The method is very close to that of the construction of stochastic integrals. One considers first piecewise constant in time integrands, as follows

$$\text{(3.5)}\qquad e_*^N(t;\omega) = \sum_{k=0}^{N-1} e_*^{N,k}(\omega)\mathbb{1}_{(\frac{kT}{N},\frac{(k+1)T}{N})}(t)$$

where $e_*^{N,k}(\omega)$ is a random variable with values in E', which is $\mathcal{E}^{\frac{kT}{N}}$ measurable. Thanks to this property $e_*^N(t)$ is adapted to the filtration $\mathcal{E}^t$. Moreover we assume

$$E \left\| e_*^{N,k} \right\|^2 < \infty \tag{3.6}$$

which implies (3.2). We set

$$\xi_{e_*^N(\cdot;\omega)}(\omega) = \sum_{k=0}^{N-1} \xi_{e_*^{N,k}(\omega)\, \mathbb{1}_{(\frac{kT}{N},(k+1)\frac{T}{N})}}(\omega). \tag{3.7}$$

The random variable $\xi_{e_*^{N,k}(\omega)\, \mathbb{1}_{(\frac{kT}{N},\frac{(k+1)T}{N})}}(\omega)$ can be defined by composition of applications, or by making use of linearity

$$\xi_{e_*^{N,k}(\omega)\, \mathbb{1}_{(\frac{kT}{N},(k+1)\frac{T}{N})}}(\omega) = \sum_j \Big((e_*^{N,k}(\omega), e_{*,j}) \Big)\, \xi_{e_{*,j}\, \mathbb{1}_{(\frac{kT}{N},(k+1)\frac{T}{N})}}(\omega) \tag{3.8}$$

where $e_{*,j}$ is an orthonormal basis of E', provided of course, we show that this limit exists (in fact in $L^2(\Omega, \mathcal{A}, P)$). We skip this step, but the following calculation shows implicitly that this limit is valid.
Then one notices that

$$E\, \xi_{e_*^{N,k}(\omega)\, \mathbb{1}_{(\frac{kT}{N},(k+1)T\frac{T}{N})}}(\omega)\, \xi_{e_*^{N,\ell}(\omega)\, \mathbb{1}_{(\frac{\ell T}{N},(\ell+1)\frac{T}{N})}}(\omega) = 0, \text{ if } k \neq \ell. \tag{3.9}$$

Indeed assume to fix the ideas $\ell \geq k+1$, then note that $\xi_{e_*^{N,k}(\omega)\mathbb{1}_{(\frac{kT}{N},(k+1)\frac{T}{N})}}(\omega)$ is $\mathcal{E}^{\ell\frac{T}{N}}$ measurable, and

$$E\, \left[\xi_{e_*^{N,\ell}(\omega)\mathbb{1}_{(\ell\frac{T}{N},(\ell+1)\frac{T}{N})}}(\omega) \middle|\, \mathcal{E}^{\ell\frac{T}{N}} \right] \;=\; 0$$

since $e_*^{N,\ell}(\omega)$ is $\mathcal{E}^{\ell\frac{T}{N}}$ and $\xi_{e_*\mathbb{1}_{(\frac{\ell T}{N},(\ell+1)\frac{T}{N})}}(\omega)$ is independant of $\mathcal{E}^{\ell\frac{T}{N}}$, for any deterministic e_*, thanks to the assumption (3.1). Hence (3.9) follows. Consider next

$$E\, \left(\xi_{e_*^{N,k}\, \mathbb{1}_{(\frac{kT}{N},(k+1)\frac{T}{N})}}(\omega) \right)^2 = E\Big[[E\big(\quad \big)^2 \big| \mathcal{E}^{\frac{kT}{N}}] \Big]$$

$$= E\, \left[E\Big(\xi_{e_*\, \mathbb{1}_{(\frac{kT}{N},(k+1)\frac{T}{N})}}(\omega) \Big)^2 \Big|_{e_*=e_*^{N,k}(\omega)} \right]$$

$$= E\, \int_{k\frac{T}{N}}^{(k+1)\frac{T}{N}} \;<\; Q(\tau) e_*^{N,k}(\omega),\; e_*^{N,k}(\omega) > d\tau\ .$$

Therefore we have shown that

$$E\left(\xi_{e_*^N(\cdot;\omega)}(\omega)\right)^2 = E\int_0^T < Q(\tau)e_*^N(\tau,\omega),\ e_*^N(\tau;\omega) > d\tau\ .$$

From this property and classical reasonings related to stochastic integrals, one obtains (3.4). □

We check further properties, which generalize similar ones related to stochastic integrals. We have

Proposition 3.2 : *The following properties hold*

$$(3.10)\qquad E\left[\xi_{e_*(\cdot;\cdot)1_{(s,t)}}(\cdot)\Big|\mathcal{E}^s\right]=0$$

$$(3.11)\ E\left[\left(\xi_{e_*(\cdot;\cdot)1_{(s,t)}}(\cdot)\right)^2\Big|\mathcal{E}^s\right]=E\left[\int_s^t < Q(\tau)e_*(\tau,\omega),e_*(\tau;\omega) > d\tau\Big|\mathcal{E}^s\right]$$

Proof We shall concentrate on (3.10). The proof of (3.11) is similar. Consider first (3.5), and the random variable $\xi_{e_*^N(\cdot,\omega)1_{(s,t)}}(\omega)$ with $s \le t$. Clearly

$$\xi_{e_*^N(\cdot,\omega)1_{(s,t)}}(\omega)=\sum_{k=0}^{n-1}\xi_{e_*^{N,k}(\omega)1_{(\frac{kT}{N}\vee s,\frac{(k+1)T}{N}\wedge t)}}(\omega)$$

and the interval $(\frac{kT}{N}\vee s, \frac{(k+1)T}{n}\wedge t)$ is not void when $s<\frac{(k+1)T}{N}$ and $t>\frac{kT}{N}$. Consider then

$$E\left[\xi_{e_*^N(\cdot,\omega)1_{(s,t)}}(\omega)\Big|\mathcal{E}^s\right]=$$

$$\sum_{k=0}^{n-1}E\left[\xi_{e_*^{N,k}(\omega)1_{(\frac{kT}{N}\vee s,\frac{(k+1)T}{n}\wedge t)}}\Big|\mathcal{E}^s\right]$$

$$=\sum_{k=0}^{N-1}E\left[E\left[\xi_{e_*^{N,k}(\omega)1_{(\frac{kT}{N}\vee s,\frac{(k+1)T}{N}\wedge t)}}\Big|\mathcal{E}^{\frac{kT}{N}\vee s}\right]\Big|\mathcal{E}^s\right].$$

Noting that $e_*^{N,k}(\omega)$ is $\mathcal{E}^{\frac{kT}{N}\vee s}$ measurable, and using again the 2nd part of assumption (3.1), we have

$$E\xi_{e_*^{N,k}(\omega)1_{(\frac{kT}{N}\vee s,\frac{(k+1)T}{n}\wedge t)}}\Big|\mathcal{E}^{\frac{kT}{N}\vee s}\Big]=0.$$

Collecting results, we deduce

$$E\left[\xi_{e_*^N(;\omega 1_{[s,t)}^{(\omega)}}\middle|\mathcal{E}^s\right]=0$$

and thus (3.10 follows easily. □

We can then introduce the stochastic process

$$I(t;\omega)=\xi_{e_*(;\omega)1_{(0,t)}}(\omega) \tag{3.12}$$

which is adapted to the filtration $\mathcal{E}^t$, and thanks to (3.10) is an $\mathcal{E}^t$ martingale. Noting that each process

$$w_i(t)=\xi_{e_{*,i}1_{(0,t)}}$$

is a Wiener process, we can up to an <u>equivalence</u>, consider $w_i(t)$ to be continuous. We can then check the

Proposition 3.2 it Up to an equivalence the process $I(t;\omega)$ is continuous.
<u>Proof</u>. We shall first prove that

$$I^N(t;\omega)=\xi_{e^N_{*,i}1_{(0,t)}}(\omega) \tag{3.13}$$

is a continuous ϵ^t martingale. It is sufficient to prove that, $\forall k$

$$\xi_{e_*^{N,k}(\omega)\mathbf{1}_{(\frac{kT}{N},\frac{(k+1)}{N}T\wedge t)}}(\omega \quad \text{is an } \mathcal{E}^t \text{ continuous martingale.} \tag{3.14}$$

Define

$$P^m e_*^{n,k}(\omega)=\sum_{j=1}^{m}\Big((e_*^{n,k}(\omega),e_*,j)\Big)e_{*,j}$$

which is the projection of $e_*^{N,k}(\omega)$ on the finite dimensional vector space spanned by $e_{*,1},\dots,e_{*,m}$. Then, from the continuity of $w_i(t)$, and the definition (3.8) we can easily assert that

$$\xi_{P^m e_*^{N,k}(\omega)\mathbf{1}_{(\frac{kT}{N},\frac{(k+1)}{N}T\wedge t)}}(\omega) \quad \text{is an } \mathcal{E}^t \quad \text{continuous martingale.} \tag{3.15}$$

Therefore, from classical properties of martingales, it follows that

$$\begin{aligned} E\sup_{0\le t\le T}\left|\xi_{P^m e_*^{N,k}1_{(\frac{kT}{N},\frac{(k+1)}{N}T\wedge t)}}\right|^2 &\le 4E\int_{\frac{kT}{N}}^{\frac{(k+1)}{N}T} < Q(\tau)P^m e_*^{N,k}, P^m e_*^{N,k} > d\tau \\ &\le c_N E\left\|P^m e_*^{N,k}\right\|^2. \end{aligned} \tag{3.16}$$

We can proceed with a classical reasoning. Define the subsequence P^{m_n} such that

$$E\left\| P^{m_n} e_*^{N,k} - e_*^{N,k} \right\|^2 \leq \frac{1}{2^n}$$

then

$$E\left\| P^{m_n} e_*^{N,k} - P^{m_{n+1}} e_*^{N,k} \right\|^2 \leq \frac{3}{2^n}$$

Therefore, from (3.16)

$$E \sup_{0\leq t\leq T} \left| \xi_{P^{m_n} e_*^{N,k} 1_{(\frac{kT}{N}, \frac{(k+1)T}{N})}(s)} - \xi_{P^{m_{n+1}} e_*^{N,k} 1_{(\frac{kT}{N}, \frac{(k+1)T}{N})}(s)} \right|^2 \leq C_N \frac{3}{2^n}$$

and

$$P\Big(\sup_{0\leq t\leq T} \left| \xi_{P^{m_n} e_*^{N,k} 1_{(\frac{kT}{N}, \frac{(k+1)T}{N})}(s)} - \xi_{P^{m_{n+1}} e_*^{N,k} 1_{(\frac{kT}{N}, \frac{(k+1)T}{N})}(s)} \right| > \frac{1}{n^2} \Big) \leq 3 C_N \frac{n^4}{2^n}.$$

On the basis of Borel Cantelli Lemma, since the series $\frac{n^4}{2^n}$ is convergent, we deduce, that a.s., $\exists n(\omega)$ such that $n \geq n(\omega)$ implies

$$\sup_{0\leq t\leq T} \left| \xi_{P^{m_n} e_*^{N,k} 1_{(\frac{kT}{N}, \frac{(k+1)T}{N})}(s)} - \xi_{P^{m_{n+1}} e_*^{N,k} 1_{(\frac{kT}{N}, \frac{(k+1)T}{N})}(s)} \right| \leq \frac{1}{n^2}$$

and thus a.s.,

$$\xi_{P^{m_n} e_*^{N,k} 1_{(\frac{kT}{N}, \frac{(k+1)T}{N})}(s)} \quad \text{converges uniformly in } t,$$

towards a limit which is continuous, as $n \to +\infty$. The property (3.14) follows, up to an equivalence. The result is thus true for (3.13).

Since for each $e_*(.)$ adapted to $\mathcal{E}^t$ and satisfying (3.2), we can associate a sequence e_*^N as in (3.5), with

$$E \int_0^T \|e_*^N - e_*\|^2 dt \to 0$$

We can perform again a reasoning similar to that above to complete the proof. □

Proposition 3.3. Let τ_1, τ_2 be two $\mathcal{E}^t$ stopping times with $0 \leq \tau_1 \leq \tau_2 \leq T$ a.s. when one has

$$E\Big[\xi_{e_*(\cdot);\cdot)\mathbf{1}_{(\tau_1,\tau_2)}}(\cdot)\Big|\,\mathcal{E}^{\tau_1}\Big] = 0 \quad \text{a.s.} \tag{3.17}$$

$$\begin{aligned} &E\Big[\Big(\xi_{e_*(\cdot);\cdot)1_{(\tau_1,\tau_2)}}(\cdot)\Big)^2\Big|\,\mathcal{E}^{\tau_1}\Big] = \\ &E\Big[\int_{\tau_1}^{\tau_2} < Q(\tau)e_*(\tau,\omega), e_*(\tau;\omega > d\tau\Big|\,\mathcal{E}^{\tau_1}\Big] \quad \text{a.s.} \end{aligned} \tag{3.18}$$

<u>Proof</u>. (3.17) is equivalent to

$$E\Big[I(\tau_2)|\mathcal{E}^{\tau_1}\Big] = I(\tau_1)$$

which follows from Doob's optional sampling Theorem, since $I(t)$ is a continuous $\mathcal{E}^t$ martingale, and τ_1, τ_2 are bounded.
Similarly (3.11) implies that

$$\Big(I(t)\Big)^2 - \int_0^t < Q(\tau)e_*(\tau), e_*(\tau) > d\tau \quad \text{is an} \quad \mathcal{E}^t \quad \text{martingale.}$$

Using again Doob's optional sampling Theorem, the property (3.18) follows easily. □

Finally, we extend the concept $\xi_{e_*(\cdot;\omega)}(\omega)$ to processes $e_*(t;\omega)$ which are adapted to $\mathcal{E}^t$, with values in E', and instead of (3.2) satisfy

$$\int_0^T \|e_*(t)\|^2 dt < \infty \quad \text{a.s.} \tag{3.19}$$

We set (not to be confused with (3.5))

$$e_*^N(t;\omega) = \left| \begin{array}{ll} e_*(t;\omega) & \text{if} \quad \int_0^t \|e_*(s)\|^2 ds \leq N \\ 0 & \text{if} \quad \int_0^t \|e_*(s)\|^2 ds > N \end{array} \right. \tag{3.20}$$

then

$$\int_0^T \|e_*^N(t)\|^2 dt \;=\; \int_0^{t_N} \|e_*(t)\|^2 dt$$

where

$$t_N = \sup\Big\{t \le T \Big| \int_0^t \|e_*(s)\|^2 ds \ \le \ N\Big\}$$

and

$$\int_0^T \|e_*^N(t)\|^2 dt \ \le \ N \tag{3.21}$$

We can consider the sequence $\xi_{e_*^N(\cdot;\cdot)}(\cdot)$, which is well defined, thanks to(3.20).
Let next

$$\Omega_N = \Big\{\omega \Big| \int_0^T \|e_*(t)\|^2 dt \ \le \ N\Big\}$$

then Ω_N is increasing and $P(\Omega - \bigcup_N \Omega_N) = 0$.
Let $\omega \in \bigcup_N \Omega_N$, in particular $\Omega \in \Omega_N$, $\forall N \ge N_0(\omega)$. Clearly

$$e_*^N(\cdot;\omega) = e_*(\cdot;\omega)\ , \quad \forall n \ge N_0(\omega)$$

therefore

$$\xi_{e_*^N(\cdot;\omega)}(\omega) = \xi_{e_*^{N_0}(\cdot;\omega)} \ \forall N \ge N_0(\omega).$$

We can deduce the estimate

$$P\Big(\sup_{0\le t\le T} |\xi_{e_*(\cdot;\omega)1_{(0,t)}}(\omega)| > c\Big) \le P\Big(\int_0^T \|e_*(t)\|^2 dt \ge N\Big) + \frac{N}{c^2} \tag{3.22}$$

Indeed, by the martingale property

$$P\Big(\sup_{0\le t\le T} |\xi_{e_*^N(\cdot;\omega)1_{(0,t)}}(\omega)| > c\Big) \le \frac{1}{c^2} E \int_0^T \|e_*^N(t)\|^2 dt \le \frac{N}{c^2}$$

and

$$\xi_{e_*(\cdot;\omega)1_{(0,t)}}(\omega) = \xi_{e_*^N(\cdot;\omega)1_{(0,t)}}(\omega) \quad \text{if} \quad \int_0^T \|e_*(t)\|^2 dt \le N$$

which implies (3.22). It then follows that

$$\int_0^T \|e_*^N(t) - e_*(t)\|^2 dt \xrightarrow{P} 0 \,\text{implies} \tag{3.23}$$

$$\sup_{0\le t\le T} |\xi_{e_*^N(\cdot;\omega)1(0,t)} - \xi_{e_*(\cdot;\omega)1_{(0,t)}}| \xrightarrow{P} 0, \quad \text{as} \ \ N \to \infty.$$

4 Generalized Ito's formula.

Generalizing the standard definition we shall say that the scalar process β_t, adapted to $\mathcal{E}^t$ and continuous has a Ito differential, whenever we can write

$$\beta(t) = \beta_0 + \int_0^t \alpha(\tau;\omega)d\tau + \xi_{e_*(\cdot;\omega)1_{(0,t)}}(\omega) \tag{4.1}$$

where

$$\beta_0 \quad \text{is } \epsilon^0 \quad \text{measurable}, \quad E\beta_0^2 < \infty \tag{4.2}$$

$$\alpha(t) \quad \text{is adapted to } \mathcal{E}^t \text{ and } E\int_0^T |\alpha(\tau)|^2 d\tau < \infty \tag{4.3}$$

$$\begin{aligned} &e_*(t;\omega) \text{ is a stochastic process with values in } E', \\ &\text{adapted to } \mathcal{E}^t, \quad \text{such that } E\int_0^T \|e_*(t)\|^2 dt < \infty. \end{aligned} \tag{4.4}$$

Let now $\psi(x,t)$ be a $C^{2,1}$ function, we can state the following

Theorem 4.1 . *Asume (4.1) to (4.4), then the process $\psi(\beta(t),t)$ has a Ito differential given by*

$$\begin{aligned} \psi(\beta(t),t) = \psi(\beta(0),0) \quad &+ \int_0^t \Big[\frac{\partial\psi}{\partial t}(\beta(\tau),\tau) + \psi'(\beta(\tau),\tau)\alpha(\tau) \\ &+ \frac{1}{2}\psi''(\beta(\tau),\tau) < Q(\tau)e_*(\tau), e_*(\tau) > \Big]d\tau \\ &+ \xi_{\psi'(\beta(\cdot),\cdot)e_*(\cdot)\mathbf{1}_{(0,t)}} \end{aligned} \tag{4.5}$$

Remark 4.1. It is clear that the right hand side of (4.5) is of the form (4.1). □

Proof of Theorem 4.1.

To simplify we limit ourselves to the case $\beta_0 = 0$, $alpha(\tau) = 0$, $\psi(x,t) = \psi(x)$. We thus have to prove

$$\psi(\beta(t)) = \frac{1}{2}\int_0^t \psi''(\beta(\tau)) < Q(\tau)e_*(\tau), e_*(\tau) > d\tau + \xi_{\psi'(\beta(\cdot))e_*(\cdot)\mathbf{1}_{(0,t)}} \tag{4.6}$$

We can next assume that

$$\|e_*(t;\omega)\| \le c. \tag{4.7}$$

Indeed, suppose (4.6) is proved whenever (4.7) holds, then set

$$e_*^n(t;\omega) = \left| \begin{array}{ll} e_*(t;\omega) & \text{if} \quad \|e_*(t;\omega)\| \le n \\ \dfrac{n e_*(t;\omega)}{\|e_*(t;\omega)\|} & \text{if} \quad \|e_*(t;\omega)\| > n \end{array} \right.$$

and

$$\beta^n(t) = \xi_{e_*^n(\cdot;\cdot) \mathbb{1}_{(0,t)}}.$$

Since

$$\text{a.s.} \qquad e_*^n \to e_* \quad \text{in} \quad L^2(0,T,E')$$

we deduce from (3.23) that

$$\sup_{0 \le t \le T} |\beta^n(t) - \beta(t) vert \xrightarrow{P} 0$$

By extracting a subsequence, we can assert that

$$\text{a.s.} \qquad \sup_{0 \le t \le T} |\beta^n(t) - \beta(t)| \to 0$$

One can then pass to the limit and obtain (4.6). We can then assume (4.7). Let then

$$\tau_n = \sup \Big\{ 0 \le t \le T \ ||\beta(t)| > n \Big\}$$

clearly $\tau_n \uparrow T$ a.s. . Suppose we prove (4.6) with $\beta(t)$ changed into $\beta^n(t) = \beta(t \wedge \tau_n)$, which corresponds to changing $e_*(\cdot)$ into $e_*^n(\cdot) = e_*(\cdot) \mathbb{1}_{(0,\tau_n)}$, then leting $n \uparrow \infty$, we deduce easily (4.6). Therefore we may as well assume that

$$|\beta(t;\omega)| \le c. \tag{4.8}$$

We thus assume (4.7), (4.8) and prove (4.6) for $t = T$. Hence

$$\psi(\beta(T)) = \frac{1}{2} \int_0^T \psi''(\beta(t)) < Q(t) e_*(t), e_*(t) > dt + \xi_{\psi'(\beta(\cdot)) e_*(\cdot)} \tag{4.9}$$

We write

$$\psi(\beta(T)) = \sum_{k=0}^{N-1} \psi\Big(\beta \frac{(k+1)T}{N}\Big) - \psi\Big(\beta(\frac{kT}{N})\Big)$$

$$= \sum_{k=0}^{N-1} \psi'\Big(\beta(\frac{kT}{N})\Big)\Big(\beta(\frac{(k+1)T}{N}) - \beta(\frac{kT}{N})\Big)$$

$$+ \sum_{k=0}^{N-1} \int_0^1 \int_0^1 \lambda \psi''\Big(\beta(\frac{kT}{N}) + \lambda\mu(\beta(\frac{(k+1)T}{N}) - \beta(\frac{kT}{N})\Big)\Big(\beta(\frac{(k+1)T}{N}) - \beta\frac{(kT)}{N}\Big)^2 d\lambda d\mu$$

$$= I_N + II_N$$

$I_N = \xi_{\sum_{k=0}^{N-1} \psi'(\beta(\frac{kT}{N}))1_{(\frac{kT}{N}, \frac{(k+1)T}{N})} e_*(\cdot)}$

We have, noting (4.7)

$$\text{a.s.} \int_0^T |\psi'(\beta(t)) - \sum_{k=0}^{N-1} \psi'(\beta(\frac{kT}{N})) 1\!\!1_{(\frac{kT}{N}, \frac{(K+1)T}{N})}|^2 \|e_*(t)\|^2 dt \to 0.$$

Therefore, using (3.23), we can assert that

$$I_N \overset{P}{\to} \xi_{\psi'(\beta(\cdot))e_*(\cdot)} \tag{4.10}$$

Next

$$II_N = \frac{1}{2} \sum_{k=0}^{N-1} \psi''\Big(\beta(\frac{(k+1)T}{N}) - \beta(\frac{kT}{N})\Big)^2 + II'_N.$$

Set

$$Z_N = \sum_{k=0}^{N-1} \Big(\beta\frac{(k+1)}{N}\Big) - \beta(\frac{kT}{N})\Big)^2$$

$$Y_N = \sup_{k=0,\cdots N-1} \int_0^1 \int_0^1 \lambda \Big|\psi''(\beta(\frac{kT}{N}) + \lambda\mu(\beta(\frac{(k+1)T}{N}) - \beta(\frac{kT}{N}))) + -\psi''\Big(\beta(\frac{kT}{N})\Big)\Big| d\lambda d\mu$$

then

$$|II'_N| \le Y_N Z_N$$

and

$$\begin{aligned} P\Big(Y_N Z_N \ge \epsilon\Big) &\le P(Z_N \ge \delta) + P(Y_N \ge \frac{\epsilon}{\delta}) \\ &\le \frac{1}{\delta} E \int_0^T < Q(\tau)e_*(\tau), e_*(\tau) > d\tau + P(Y_N \ge \frac{\epsilon}{\delta}). \end{aligned}$$

But, from the continuity of $\beta(t)$,

$$Y_N \to 0 \quad \text{a.s. , as } N \to 0$$

hence

$$P(Y_N \geq \frac{\epsilon}{\delta}) \to 0 \quad \text{as } N \to \infty, \ \forall \epsilon, \delta.$$

It follows that

$$II'_N \xrightarrow{P} 0 \quad \text{as } N \to \infty \tag{4.11}$$

Set

$$III_N = \frac{1}{2}\sum_{k=0}^{N-1} \psi''(\beta(\frac{kT}{N}) \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} < Q(t)e_*(t), e_*(t) > dt$$
$$\to \frac{1}{2}\int_0^T \psi''(\beta(t)) < Q(t)e_*(t), e_*(t) > dt$$

The proof will be completed if we show that

$$\Sigma_N = \sum_{k=0}^{N-1} \psi''(\beta\frac{kT}{N})\Big[\Big(\beta\frac{(k+1)T}{N} - \beta(\frac{kT}{N})\Big)^2 + - \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} < Q(t)e_*(t), e_*(t) > dt\Big] \xrightarrow{P} 0, \quad \text{as } N \to \infty. \tag{4.12}$$

Thanks to (4.7), (4.8) we can compute

$$E\Sigma_N^2 = \sum_{k=0}^{N-1} E\Big(\psi''(\beta(\frac{kT}{N}))\Big)^2\Big[\Big(\beta(\frac{(k+1)T}{N}) - \beta(\frac{kT}{N})\Big)^2 - \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} < Q(t)e_*(t), e_*(t) > dt\Big]^2 \tag{4.13}$$

since for $k < \ell$, using (3.11) one has

$$E\psi''(\beta(\frac{kt}{n}))\psi''(\beta(\frac{\ell T}{N})\Big[\Big(\beta(\frac{(k+1)T}{N}) - \beta(\frac{kT}{N})\Big)^2 - \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} < Q(t)e_*(t), e_*(t) > dt\Big]$$

$$\Big[\Big(\beta(\frac{(\ell+1)T}{N})-\beta(\frac{\ell T}{N})\Big)^2-\int_{\frac{\ell T}{N}}^{\frac{(\ell+1)T}{N}} <Q(t)e_*(t),e_*(t)>dt\Big]=0.$$

Then

$$\begin{aligned} E\Sigma_N^2 &\le c\sum_{k=0}^{N-1}E\Big[\Big(\beta(\frac{(k+1)T}{N})-\beta(\frac{kT}{N})\Big)^2-\int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} <Q(t)e_*(t),e_*(t)>dt\Big]^2 \\ &\le c\sum_{k=0}^{N-1}E\Big[\beta(\frac{(k+1)T}{N})-\beta(\frac{kT}{N})\Big]^4+\frac{c}{N}. \end{aligned}$$

But

$$\sum_{k=0}^{N-1}E\Big[\beta(\frac{(k+1)T}{N})-\beta(\frac{kT}{N})\Big]^4\le E\Big\{\sup_{k=0,\cdots N-1}\Big|\beta(\frac{(k+1)T}{N})\Big)-\beta(\frac{kT}{N})\Big|^2$$

$$\sum_{k=0}^{N-1}\Big(\beta(\frac{(k+1)T}{N})-\beta(\frac{kT}{N})\Big)^2\Big\}$$

$$\le\Big(E\sup_{k=0\cdots N-1}\Big|\beta(\frac{(k+1)T}{N})-\beta(\frac{kT}{N})\Big|^4\Big)^{1/2}$$

$$\Big(E\Big[\sum_{k=0}^{N=1}(\beta\frac{(k+1)T}{N})-\beta(\frac{kT}{N}))\Big]^2\Big)^{1/2}$$

Thanks to (4.8)

$$E\sup_{k=0\cdots N-1}\Big|\beta(\frac{(k+1)T}{N}-\beta(\frac{kT}{N})\Big|^4\to 0,\quad \text{as}\ \ N\to+\infty. \tag{4.14}$$

Set next

$$V_\ell=\sum_{k=0}^{\ell-1}\Big(\beta(\frac{(k+1)T}{N})-\beta(\frac{(kT)}{N})\Big)^2,\quad \ell=1,\cdots,N$$

we have

$$V_N^2=\sum_{k=0}^{N-1}\Big(\beta(\frac{(k+1)T}{N})-\beta(\frac{(kT)}{N})\Big)^4=2\sum_{k=0}^{N-1}\Big(\beta(\frac{(k+1)T}{N})-\beta(\frac{kT}{N})\Big)^2(V_N-V_{k+1})$$

But

$$E\Big[V_N - V_{k+1}\Big\|\mathcal{E}^{\frac{(K+1)T}{N}}\Big] = \sum_{j=k+1}^{N-1} E\Big[\Big(\beta(\frac{(k+1)T}{N}) - \beta(\frac{kT}{N})\Big)^2\Big|\mathcal{E}^{\frac{(k+1)T}{N}}\Big]$$

$$= \sum_{j=k+1}^{N-1} E\Big[\int_{\frac{jT}{N}}^{\frac{(j+1)T}{N}} < Q(\tau)e_*(\tau), e_*(\tau) > dt\Big|\mathcal{E}^{\frac{(k+1)T}{N}}\Big] \leq C.$$

Therefore, as easily seen

$$EV_N^2 \leq C$$

hence, collecting results one gets

$$E\Sigma_N^2 \to 0, \quad \text{as } \; N \to \infty$$

which proves in particular (4.12) and completes the proof. □

We can generalize (4.5) to functions $\psi(x,t)$, $x \in R^m$, and process $\beta(t) = (\beta^1(t), \cdots, \beta^m(t)$, with

$$\beta^i(t) = \beta_0^i + \int_0^t \alpha^i(\tau;\omega)d\tau + \xi_{e_*^i(\cdot;\omega)1_{(0,t)}}(\omega)$$

We have

$$\psi(\beta(t),t) = \psi(\beta_0,0) + \int_0^t \Big[\frac{\partial\psi}{\partial t}(\beta(\tau),\tau) + \Sigma_i \frac{\partial\psi}{\partial x_i}(\beta(\tau),\tau)\alpha^i(\tau;\omega)$$

$$+\frac{1}{2}\Sigma_{i,j}\frac{\partial^2\psi}{\partial x_i\partial x_j}(\beta(\tau),\tau) < Q(\tau)e_*^i(\tau), e_*^j(\tau) > d\tau\Big]d\tau \tag{4.15}$$

$$+\xi_{\Sigma_i \frac{\partial\psi}{\partial x_i}(\beta(\cdot),\cdot)e_*^i(\cdot)1_{(0,t)}}$$

5 Linear stochastic evolution equations

5.1 Notation - Statement of the results

Let us consider the usual triple

$$V \subset H \subset V'$$

which is a sequence of Hilbert spaces, each space being continuously embedded in the next one, H is identical to its dual, and V' is the dual of V. All spaces are separable.
Let $A(\cdot) \in L^\infty(0,T;\mathcal{L}(V;V'))$, satisfying the coercity property

$$< A(t)v, v > \geq \alpha \|v\|^2, \quad \forall v \in V, \ \alpha > 0. \tag{5.1}$$

Let next E be another Hilbert space and

$$B(\cdot) \in L^\infty(0,T;\mathcal{L}(E;V')). \tag{5.2}$$

We shall consider two Linear Random Functionals, as follows $\xi_h(\omega)$, on H, and $\xi_{e_*(\cdot)}(\omega)$ on $L^2(0,T;E')$ which are gaussian, independant from each other, with mathematical expectation 0, and covariance operator $P_0 \in \mathcal{L}(H;H)$ and $Q(\cdot) \in L^\infty(0,T;\mathcal{L}(E';E))$ as in (2.10).
We are going to prove the following.

Theorem 5.1. *There exists a unique L.R.F. $y_{\varphi_*(\cdot)}(\omega)$ on $L^2(0,T,V')$, and a unique family $y_h(t;\omega)$ of L.R.F. on H, indexed by t, such that $y_h(t) \in L^\infty(0,T;\mathcal{L}(H;L^2(\Omega,\mathcal{A},P)$ and $\int_0^T y_{\varphi_*(t)}(t;\omega)dt$ can be extended from $L^2(0,T;H)$ to $L^2(0,T;V')$, with the relation*

$$y_{\varphi_*(\cdot)}(\omega) = \int_0^T y_{\varphi_*(t)}(t;\omega)dt \ a.s. \tag{5.3}$$

Moreover, one has the relation

$$\begin{gathered} y_h(t) + \int_0^t y_{A^*(\tau)h}(\tau)d\tau = \xi_h + \xi_{B^*(\cdot)h\mathbf{1}_{(0,t)}} \quad \text{a.s.} \\ \forall h \in V, \quad \forall t, \quad \text{a.s.} \end{gathered} \tag{5.4}$$

Moreover, if one has defines for any $\varphi_* \in L^2(0,T;V')$ the function $p \in L^2(0,T;V)$, $p' \in L^2(0,T;V')$ such that

$$\begin{gathered} -p' + A^*(t)p = \varphi_*(t) \quad \text{a.e.} \\ p(T) = 0 \end{gathered} \tag{5.5}$$

and for any $h \in H$, the family $q_t(\tau) \in L^2(0,T;V)$, $q_t' \in L^2(0,T;V')$ such that

$$-q_t' + A^*(\tau) q_t = 0 \qquad q_t(t) = h \tag{5.6}$$

then one has the relations

$$y_h(t) = \xi_{q_t(0)} + \xi_{B^* q_t(\cdot) 1_{(0,t)}} \tag{5.7}$$

$$y_{\varphi_*(\cdot)} = \xi_{t(0)} + \xi_{B^* p(\cdot)} \tag{5.8}$$

5.2 Proof of Theorem 5.1

.

We use the Galerkin method. Let $h_1, \cdots, h_m \cdots$ be an orthonormal basis of H, made of elements of V.

We define, for fixed m the scalar processes $y_i^m(t)$, $i = 1 \cdots m$, solution of the integral equations

$$y_i^m(t) + \int_0^t \sum_{j=1}^m < A(\tau) h_j, h_i > y_j^m d\tau = \xi_{h_i} + \xi_{B^*(\cdot) h_i 1_{0,t)}} . \tag{5.9}$$

Let $\mathcal{E}^t = \sigma(\xi_h, \xi_{e_*(\cdot) 1_{(0,t)}}, \forall h \in H,\ e_*(\cdot) \in L^2(0,T;E'))$, then the processes $y_i^m(t)$ are adapted to $\mathcal{E}^t$. We then define a L.R.F. on H, by the formula

$$y_h^m(t) = \sum_{i=1}^m y_i^m(t)(h, h_i), \tag{5.10}$$

and if $\varphi_*(\cdot) \in L^2(0,T;V')$, we set

$$y_{\varphi_*}^m = \int_0^T \sum_{i=1}^m (t) y_i^m(t) < \varphi_*(t), h_i > dt . \tag{5.11}$$

Let us introduce the detrministic function $p_i^m(t)$ solutions of the differential system

$$\begin{aligned} &-(p_i^m(t))' + \sum_{j=1}^m < A^*(\tau) h_j,\ h_i > p_j^m = < \varphi_*(t), h_i > \\ &p_i^m(T) = 0, \quad i = 1 \cdots m \end{aligned} \tag{5.12}$$

Apply Ito's formula (4.15) to the process $\sum_{i=1}^{m} y_i^m(t) p_i^m(t)$ we obtain

$$
\begin{aligned}
\sum_{i=1}^{m} y_i^m(t) p_i^m(t) = \xi_{p^m(0)} + \int_0^t \Big[\sum_{i=1}^{m} y_i^m(\tau)(p_i^m(\tau))' \\
- \sum_{i=1}^{m} p_i^m(\tau) \sum_{j=1}^{m} < A(\tau) h_j, h_i > y_j^m(\tau) \Big] d\tau \\
+ \xi_{B^* p_m(\cdot) 1_{(0,t)}}
\end{aligned} \tag{5.13}
$$

where we have set

$$
p^m(t) = \sum_{i=1}^{m} p_i^m(t) h_i \; .
$$

Thanks to (5.12), (5.11) we deduce

$$
y_{\varphi_*}^m = \xi_{p^m(0)} + \xi_{B^* p^m(\cdot)} \tag{5.14}
$$

Similarly, if we define

$$
q_t^m(\tau) = \sum_{i=1}^{m} q_{t,i}^m(\tau) h_i
$$

with

$$
\begin{aligned}
-(q_{t,i}^m)' + \sum_{j=1}^{m} < A^*(\tau) h_j, h_i > q_{t,j}^m = 0, \quad \tau < t \\
q_{t,i}^m(t) = (h, h_i)
\end{aligned} \tag{5.15}
$$

and proceeding as in (5.13) yields

$$
y_h^m(t) = \xi_{q_t^m(0)} + \xi_{B^* q_t^m(\cdot) 1_{(0,t)}} . \tag{5.16}
$$

Therefore, one obtain immediately

$$
E|y_h^m(t)|^2 = (P_0 q_t^m(0), q_t^m(0)) + \int_0^t < BQB^* q_t^m(\tau), q_t^m(\tau) > d\tau \tag{5.17}
$$

$$
E|y_{\varphi_*}^m|^2 = (P_0 p^m(0), p^m(0)) + \int_0^t < BQB^* p^m(\tau), p^m(\tau) > d\tau \tag{5.18}
$$

for which, and classical estimates concerning (5.12), (5.15), we obtain

$$E|y^m_{\varphi_*}|^2 \leq c\int_0^T |\varphi_*(t)|^{2'}{}_{V}dt \tag{5.19}$$

$$E|y^m_h(t)|^2 \leq c|h|^2. \tag{5.20}$$

Therefore $y^m_h(t)$ remains bounded in $L^\infty(0,T;\mathcal{L}(H;L^2(\Omega,\mathcal{A},P)))$ and $y^m_{\varphi_*}$ remains bounded in $\mathcal{L}(L^2(0,T;V');L^2(\Omega,\mathcal{A},P))$. From (5.16), one checks easily that

$$\begin{aligned} y^m_h(t) &\to \xi_{\varphi(0)} + \xi_{B^*\varphi 1(0,t)} \quad \text{in} \quad L^2(\Omega,\mathcal{A},P),\ \forall t,h \\ y^m_{\varphi_*} &\to \xi_{p(0)} + \xi_{B^*p(\cdot)} \quad \text{in} \quad L^2(\Omega,\mathcal{A},P),\ \forall \varphi_* \end{aligned}$$

Denoting $y_h(t), y_{\varphi_*(\cdot)}$ the limits, we obtain (5.7), (5.8).

On the other hand, comparing (5.10), (5.11) we note that

$$y^m_{\varphi_*} = \int_0^T y^m_{\varphi_*(t)}(t)dt\ . \tag{5.21}$$

If $\varphi_* \in L^2(0,T;H)$, we can pass to the limit in the integral on the right hand side of (5.21) and obtain

$$y_{\varphi_*} = \int_0^T y_{\varphi_*(t)}(t)dt \tag{5.22}$$

Since the left hand side is well-defined as a L.R.F. on $L^2(0,T;V')$, it follows that the integral can be extended as well to $\varphi_*(\cdot) \in L^2(0,T;V')$.

It remains to prove the relation (5.4). We choose a special basis of H such that

$$((h,h_j)) = \lambda_j(h,h_j) \quad \forall h \in H, \quad \lambda_j > 0\ .$$

Note that $\frac{h_j}{\sqrt{\lambda_j}}$ is an orthonormal basis of V, and $\sqrt{\lambda j}h_j$ is an orthonormal basis of V'.

For $h \in V$, let

$$h^m = \sum_{i=1}^m h_i(h,h_i) = \sum_{i=1}^m \frac{h_i}{\sqrt{\lambda_i}}((h,\frac{h_i}{\sqrt{\lambda_i}}))$$

then $h^m \to h$ in V. Note that (5.9 can be written as

$$y^m_h(t) + \int_0^t \sum_{j=1}^m < A(\tau)h_j, h^m > y^m_j d\tau = \xi_{h^m} + \xi_{B^*(\cdot)h^m 1_{0,t)}}$$

or

(5.23) $$y_h^m(t) + \int_0^t y^m_{A^*(\tau)h^m}(\tau)d\tau = \xi_{h^m} + \xi_{B^*(\cdot)h^m 1_{(0,t)}}$$

Since

$$\begin{aligned} E\Big(\int_0^t y^m_{A^*(\tau)(h^m-h)}(\tau)(d\tau)\Big)^2 &\leq c\int_0^t \|A^*(\tau)(h^m-h)\|^2_{V'}, d\tau \\ &\leq c\|h^m-h\|^2_V \to 0 \end{aligned}$$

and

$$\int_0^t y^m_{A^*(\tau)h}(\tau)d\tau = y^m_{A^*(\cdot)h1_{(0,t)}} \to y_{A^*(\cdot)h1_{(0,t)}} = \int_0^t y_{A^*(\tau)h}(\tau)d\tau$$

$$\text{in} \quad L^2(\Omega, \mathcal{A}, P), \quad \forall t,$$

we can pass to the limit in (ref 5.23), $\forall t$, $\forall h \in V$, and obtain (5.4). Thanks to (5.7), (5.8), the uniqueness is clear. This completes the proof. □

5.3 Correlation operator

If we set

(5.24) $$\Big(\Pi(t)h, h'\Big) \;=\; Ey_h(t)y'_h(t)$$

then we are going to prove the

Theorem 5.2.

$$\Pi(\cdot) \in L^\infty(0, T; \mathcal{L}(H; H)), \Pi(t) \geq 0$$

self adjoint and

(5.25)If $\theta \in L^2(0, T; V)$, $\quad \theta' \in L^2(0, T; V'), -\dfrac{d\theta}{dt} + A^*\theta \in L^2(0, T; H)$

then $\Pi(\cdot)\theta(\cdot) \in L^2(0, T; V)$, $\frac{d}{dt}\Pi\theta \in L^2(0, T, V')$,

$$\frac{d}{dt}(\Pi\theta) + \Pi(-\frac{d\theta}{dt} + A^*\theta) + A\Pi\theta = BQB^*\theta$$

$$\Pi(0) = P_0$$

Proof .

Define the approximation

$$(\Pi^m(t)h,h') = Ey^m{}_h(t)y^{m\prime}{}_h(t).$$

From (5.9) and Ito's formula (4.15), we can write, as easily seen

$$
\begin{aligned}
&\frac{d}{dt}Ey_i^m(t)y_j^m \quad + \quad <\Pi^m(t)h_i, A^*(\tau)h_j> \\
(5.26)\qquad & \\
&\qquad + \quad <\Pi^m(t)h_j, A^*(\tau)h_i> = <BQB^*(t)h_j, h_i>
\end{aligned}
$$

Let θ as in (5.25), set

$$\frac{-d\theta}{dt} + A^*\theta = \mu \in L^2(0,T;H) \ .$$

Let

$$\mu^m(t) \ = \ \sum_{j=1}^{m}\mu_j^m(t)h_j$$

and $\theta^m = \sum_{j=1}^m \theta_j^m(t)h_j$, to be the solution of

$$(5.27)\ ) \quad \frac{-d\theta_j^m}{dt} + \sum_{k=1}^{m} <A^*(t)h_k, h_j> \theta_k^m = \mu_j^m \quad \theta_j^m(T) = \Big(\theta(t), h_j\Big) \ .$$

Set also

$$(5.28) \qquad \alpha^m(t) \ = \ \Pi^m(t)\theta^m(t)$$

then using (5.26) and (5.27), we deduce easily, after cancellation

$$(5.29) \quad \frac{d}{dt}\alpha_i^m(t) + <A(t)\alpha^m(t), h_i> = <BQB^*\theta^m, h_i> - \Big(\Pi^m\mu^m, h_i\Big)$$

hence

$$\frac{1}{2}\frac{d}{dt}|\alpha^m(t)|^2 + <A(t)\alpha^m, \alpha^m> = <BQB^*\theta^m, \alpha^m> -(\Pi^m\mu^m, \alpha^m).$$

$$\frac{1}{2}\frac{d}{dt}|\theta^m(t)|^2 + <A(t)\theta^m, \theta^m> = (\mu^m, \theta^m).$$

Taking account of initial conditions

$$\alpha^m(0) = \Pi^m(0)\theta^m(0), \quad \theta^m(T) = \sum_{j=1}^{m}(\theta(T), h_j)h_j$$

it easily follows that
α^m, θ^m remain in a bounded subset of $L^2(0,T;V)$
$\frac{d}{dt}\alpha^m$, $\frac{d}{dt}\theta^m$ remain in a bounded subset of $L^2(0,T;V')$.
Taking converging subsequences, we have

$$\theta^m \to \theta, \quad \alpha^m \to \Pi\theta \quad \text{in} \quad L^2(0,T;V) \text{ weakly}$$

$$\frac{d\theta^m}{dt} \to \frac{d\theta}{dt}, \quad \frac{d\alpha^m}{dt} \to \frac{d}{dt}\Pi\theta \quad \text{in} \quad L^2(0,T;V') \text{ weakly .}$$

Passing to the limit in (5.29), one deduces (5.25) easily. □

6 FILTERING THEORY

6.1 The model

The mathematical expectation of the L.R.F. $y_h(t)$ in (5.4) is clearly 0. So, we can complete the model as folows

$$
\begin{aligned}
y_h(t) \quad &+ \quad \int_0^t y_{A^*(\tau)h}(\tau)d\tau = \int_0^t < h, g(\tau) > d\tau + (\bar{\xi}, h) + \xi_h \\
(6.1) \qquad & \\
&+ \quad \int_0^t < h, B(\tau)\xi(\tau) > d\tau + \xi_{B^*(\cdot)h1_{(0,t)}} \quad \text{a.s.} \quad , \forall h \in V, \forall t
\end{aligned}
$$

where $g \in L^2(0,T;V')$, $\bar{\xi} \in H$, $\bar{\xi} \in L^2(0,T;E)$.
Introduce the solution $\bar{y}(t)$ of

$$(6.2) \qquad \frac{d\bar{y}}{dt} + A(t)\bar{y} = g(t) + B(t)\bar{\xi}(t) \quad \bar{y} = \bar{\xi}$$

then clearly

$$(6.3) \qquad Ey_h(t) = (\bar{y}(t), h) \ .$$

Let $\mathcal{C}(\cdot) \in L^\infty(0,T;\mathcal{L}(V;F))$, F Hilbert space. We consider a gaussian L.R.F. on $L^2(0,T;F')$, denoted $\eta_{f_*(\cdot)}(\omega)$, which is independant from ζ_h, $\xi_{e_*(\cdot)}(\omega)$, and has mathematical expectation 0, and correlation operator

$$(6.4) \qquad E\eta_{f^1_*(\cdot)}\eta_{f^2_*(\cdot)} = \int_0^T < R(t)f^1_*(t), \ f^2_*(t) > dt$$

with

$$(6.5) \qquad R(\cdot) \in L^\infty(0,T;\mathcal{L}(F';F)) \ , \quad R^{-1}(\cdot) \in L^\infty(0,T;\mathcal{L}(F;F')) \ .$$

We consider the observation L.R.F. defined by the formula

$$(6.6) \qquad Z_{f_*(\cdot)}(\omega) = \int_0^T y_{\mathcal{C}^*(t)f_*(t)}(t)dt + \eta_{f_*(\cdot)}(\omega)$$

in which the integral is well defined thanks to (5.3). We have

$$(6.7) \qquad EZ_{f_*(\cdot)} = \int_0^T < f_*(t), \mathcal{C}(t)\bar{y}(t) > dt \ .$$

From (5.8), one can write also

$$Z_{f_*(\cdot)} = \zeta_{p(0)} + \xi_{B^*p(\cdot)} + \eta_{f_*(\cdot)} + \int_0^T < \mathcal{C}(t)\bar{y}(t), f_*(t) > dt \tag{6.8}$$

where p is the solution of

$$-p' + A^*(t)p = \mathcal{C}^*(t)f_*(t) \quad p(T) = 0 \tag{6.9}$$

and thus the correlation operator of Z_{f_*} is given by

$$\begin{aligned} EZ_{f_*^1(\cdot)}Z_{f_*^2(\cdot)} - EZ_{f_*^1}E_{f_*^2} = \\ (P_0p^1(0), p^2(0)) + \int_0^T < B(t)Q(t)B^*(t)p^1(t), p^2(t) > dt \\ + \int_0^T < R(t)f_*^1(t), f_*^2(t) < dt \end{aligned} \tag{6.10}$$

where p^1, p^2 are the solutions of (6.9) corresponding to $f_* = f_*^1$ and $f_* = f_*^2$ respectively.

6.2 Estimation

Let $\mathcal{B} = \sigma(Z_{f_*}, f_* \in L^2(0, T; F'))$. We define the L.R.F. on H

$$\hat{y}_h(T) = E\Big[y_h(T)|\mathcal{B}\Big] \tag{6.11}$$

and our objective is to derive a Kalman filter for $\hat{y}_h(T)$. It is useful to observe that

$$\mathcal{B} = \sigma\Big(Z_{f_*^i 1_{(0,t)}}, \quad \forall i, \quad \forall t \leq T) \tag{6.12}$$

where f_*^i is an orthonormal basis of F'.
We begin by defining linear estimates. If $S \in \mathcal{L}(L^2(0, T); H)$ we define the L.R.F.

$$y_h^S(T) = (\bar{y}(T), h) + Z_{S^*h(\cdot)} - \int_0^T < \mathcal{C}(t) bary(t), \quad S^*h(t) > dt \ . \tag{6.13}$$

Note that

$$E\mathcal{Y}_h^S(T) = (\bar{y}(T), h) \ .$$

Let the linear estimate error be defined by

$$\epsilon_h^S(T) = y_h(T) - y_h^S(T) \tag{6.14}$$

which as a mathematical expectation equal to 0. The best linear filter is the operator $\hat{S}$, if it exists, such that

$$E\left|\epsilon_h^{\hat{S}}(T)\right|^2 \leq E\left|\epsilon_h^S(T)\right|^2, \quad \forall S, \forall h \ . \tag{6.15}$$

We shall prove that such $\hat{S}$ exists, and is unique.

6.3 The best linear filter is the solution of the filtering problem

.

We prove

Theorem 6.1 *If $\hat{S}$ satisfies (6.15), then one has*

$$\hat{y}(T) = y_h^{\hat{S}}(T) \tag{6.16}$$

Proof.

Note that from (6.13)

$$\begin{aligned} y_h^{\hat{S}+S}(T) &= (\bar{y}(T), h) + Z_{\hat{S}^*h} - \int_0^T < C(t)\bar{y}(t),\ \hat{S}^*h(t) > dt \\ &+ \ Z_{S^*h} - \int_0^T < C(t)\bar{y}(t),\ S^*h(t) > dt \end{aligned}$$

changing in (6.15), S by $\hat{S} + S$, yields

$$0 \leq E\left|Z_{S^*h} - \int_0^T < C(t)\bar{y}(t), S^*h > dt\right|^2 - 2E\epsilon_h^{\hat{S}}\, Z_{S^*h}$$

Since S is arbitrary, necessarily we have

$$E\ \epsilon_h^{\hat{S}}\ Z_{S^*h} = 0\ , \quad \forall S, \quad \forall h. \tag{6.17}$$

If $f_{\ddagger}^{i}$ is the basis of F', we define for t and h give the map $S_{h,f_{\ddagger}^{i},t} \in \mathcal{L}(L^2(0,T;F);H)$ by

$$S_{h,f_{\ddagger}^{i},t} f(\cdot) = \int_0^t < f_{\ddagger}^{i}, f > d\tau \frac{h}{|h|^2}, \quad \forall f(\cdot) \in L^2(0,T;F).$$

It is easy to check that

$$S^*_{h,f_{\ddagger}^{i},t} h \ = \ f_{\ddagger}^{i} 1\!\!1_{(0,t)} \ .$$

Applying (6.17 with $S = S_{h,f_{\ddagger}^{i},t}$ we obtain

$$E\ \epsilon_h^{\hat{S}}\ Z_{f_{\ddagger}^{i} 1_{(0,t)}} \ = \ 0.$$

Since $\epsilon_h^{\hat{S}}$ and $S_{f_{\ddagger}^{i} 1_{(0,t)}}$ are gaussian, the non correlation implies the independance. Hence $\epsilon_h^{\hat{S}}$ is independant from $Z_{f_{\ddagger}^{i} 1_{(0,t)}}$, $\forall i$, $\forall t$. Therefore $\epsilon_h^{\hat{S}}$ is independant from $\mathcal{B}$. This implies (6.16) and completes the proof. □

6.4 Computation of the best linear filter

.

From (5.6), (5.7) applied to $t = T$, and (6.8), it is easy to check that, if one introduces $\beta(\cdot)$ solution of

$$(6.18) \qquad \begin{aligned} -\frac{d\beta}{dt} + A^*(t)\beta &= -C^*(t)S^*h(t) \\ \beta(t) &= h \end{aligned}$$

then one has

$$(6.19) \qquad \epsilon_h^{S}(T) = \zeta_{\beta(0)} + \xi_{B^*\beta(\cdot)} + \eta_{S^*h(\cdot)}$$

andd thus

$$(6.20) \qquad \begin{aligned} E(\epsilon_h^{S}(T))^2 = (P_0\beta(0), \beta(0)) \ &+ \ \int_0^T < B(t)Q(t)B^*(t)\beta(t), \beta(t) > dt \\ &+ \ \int_0^T < R(t)S^*h(t), S^*h(t) > dt \end{aligned}$$

From this equality, it is standard to check that considering the coupled system

$$(6.21)\qquad \begin{aligned} &\frac{d\hat{\alpha}}{dt} + A(t)\hat{\alpha} + B(t)Q(t)B^*(t)\hat{\gamma} = 0 \qquad \hat{\alpha}(0) + P_0\hat{\gamma}(0) = 0 \\ &\frac{d\hat{\gamma}}{dt} + A^*(t)\hat{\gamma} - C(t)R^{-1}(t)C(t)\hat{\alpha} = 0 \qquad \hat{\gamma}(T) = h \end{aligned}$$

which has a unique solution, and setting $\hat{S}(\cdot) \in \mathcal{L}(L^2(0,T;F);H)$ by the relation

$$(6.22)\qquad (\hat{S}f(\cdot),h) = -\int_0^T < f(t), R^{-1}(t)C(t)\hat{\alpha}(t) > dt, \quad \forall f(\cdot) \quad \forall h$$

then $\hat{S}$ satisfies (6.15), and is unique.

To make $\hat{S}$ more explicit, we associate $f(\cdot) \in L^2(0,T;F)$, the coupled system

$$(6.23)\qquad \begin{aligned} \frac{d\hat{y}}{dt} + A(t)\hat{y} + B(t)Q(t)B^*(t)\hat{p} &= 0 \\ -\frac{d\hat{p}}{dt} + A^*(t)\hat{p} - C^*(t)R^{-1}(t)C(t)\hat{y} &= -C^*(t)R^{-1}(t)f(t) \\ \hat{y}(0) + P_0\hat{p}(0) &= 0 \\ \hat{p}(T) &= 0 \end{aligned}$$

which has a unique solution. Then

$$(6.24)\qquad \hat{S}f(\cdot) = \hat{y}(T).$$

Note also that from (6.21), one has

$$(6.25)\qquad \hat{S}^*h(t) = -R^{-1}(t)C(t)\hat{\alpha}(t).$$

6.5 Decoupling Theory

To recover the Kalman filter, we use the decoupling theory (see J.L. LIONS [5], A. BENSOUSSAN [3]). We assume here that

$$(6.26)\qquad C(\cdot) \in L^\infty(0,T;\mathcal{L}(H;F)).$$

Then there exists a unique P such that

$$P(\cdot) \in L^\infty(0,T;\mathcal{L}(H;H)),\ P(t) \geq 0, \quad \text{self adjoint.} \tag{6.27}$$

$$\begin{aligned} &\text{If } \ \theta \in L^2(0,T;V),\ \theta' \in L^2(0,T;V'),\ -\frac{d\theta}{dt} + A^*\theta \in L^2(0,T;H) \\ &\qquad \text{then } \ P\theta \in L^2(0,T;V),\ \frac{d}{dt}P\theta \in L^2(0,T;V') \end{aligned} \tag{6.28}$$

$$\begin{aligned} \frac{d}{dt}(P\theta) + P(-\frac{d\theta}{dt} + A^*\theta) + AP\theta + PC^*R^{-1}CP\theta &= BQB^*\theta \\ P(0) &= P_0 \end{aligned} \tag{6.29}$$

If we consider the system (6.21), then we have

$$\hat{\alpha}(t) = -P(t)\hat{\gamma}(t)$$

hence $\hat{\gamma}$ is the solution of

$$-\frac{d\hat{\gamma}}{dt} + (A^* + C^*R^{-1}CP)\hat{\gamma} = 0 \qquad \hat{\gamma}(T) = h \tag{6.30}$$

and from (6.25), we deduce

$$\hat{S}^*h(t) = R^{-1}C(t)p(t)\hat{\gamma}(t). \tag{6.31}$$

By analogy with (5.4), (5.6), (5.7) we derive easily that the Kalman filter $\hat{y}_h(t)$ is the solution of

$$\begin{aligned} \hat{y}_h(t) + \int_0^t \hat{y}_{(A^*+C^*R^{-1}CP)(\tau)h}(\tau)d\tau &= \\ (\bar{\xi},h) + \int_0^t < h, g(\tau) + B(\tau)\bar{\xi}(\tau) > d\tau &+ Z_{R^{-1}CP(\cdot)M_{(q,t)}} \\ \forall h \in V, \quad \forall t, \quad \text{a.s.}. \end{aligned} \tag{6.32}$$

Moreover, one easily checks also

$$E\Big|\epsilon_h^{\mathcal{B}}(T)\Big|^2 = (P(T)h, h) \tag{6.33}$$

and $P(T)$ is the correlation operator of the L.R.F. $\epsilon_h^{\mathcal{B}}(T)$.

6.6 Conditional probability

The Kalman-filter is the conditional expectation of $y_h(T)$ given the σ-algebra $\mathcal{B}$. Let us check the

Theorem 6.2 . *The conditional probability of $y_{h_1}(T), \cdots, y_{h_m}(T)$ given $\mathcal{B}$ is a gaussian with conditional mean $\hat{y}_{h_1}(T), \cdots, \hat{y}_{h_m}(T)$ and conditional correlation*

$$E\Big[\Big(y_{h_i}(T) - \hat{y}_{h_i}(t)\Big)\Big(y_{h_j}(T) - \hat{y}_{h_j}(T)\Big)\Big|\mathcal{B}\Big] = \Big(P(T)h_i, h_j\Big). \tag{6.34}$$

Proof.

We consider the Fourrier transform of the conditional probability of $y_h(T)$, namely

$$E\Big[exp\ i\lambda y_h(T)\Big|\mathcal{B}\Big] = exp\ i\lambda\hat{y}_h(T)\ E\big[exp\ i\lambda\epsilon_h(T)\Big|\mathcal{B}\Big].$$

Using the fact that $\epsilon_h(T)$ is independant of $\mathcal{B}$, we have

$$\begin{aligned} E\Big[exp\ i\lambda\epsilon_h(T)\Big|\mathcal{B}\Big] &= E\ exp\ i\lambda\epsilon_h(T) \\ &= \exp\ -\frac{1}{2}\lambda^2(P(T)h, h). \end{aligned}$$

Considering next

$$E\Big[exp\ i\sum_{k=1}^{m}\lambda_k y_{h_k}(T)\Big|\mathcal{B}\Big] = E\Big[exp\ i\ y_{\sum_{k=1}^{m}\lambda_k h_k}(T)\Big|\mathcal{B}\Big]$$

$$= exp\ i\ \sum_{k=1}^{m}\lambda_k\hat{y}_{h_k}(T)\ exp\ -\frac{1}{2}\sum_{j,k}\lambda_j\lambda_k(P(T)h_j, h_k)$$

and the result is obtained.

6.7 Innovation

Consider the L.R.F. on $L^2(0,T;F')$ defined by

$$I_{f_*}(\omega) = Z_{f_*}(\omega) - \int_0^T \hat{y}_{C^*(t)f_*(t)}(t)dt \tag{6.35}$$

then one has the property

Theorem 6.3 *The L.R..F $I_{f_*}(\omega)$, called the innovation is $\mathcal{B}$ measurable, gaussian, with*

$$E\ I_{f_*} = 0 \tag{6.36}$$

$$E\ I_{f_*^1}\ I_{f_*^2} = \int_0^T < R(t)f_*^1(t),\ f_*^2(t) > dt \tag{6.37}$$

Proof. The fact that I_{f_*} is $\mathcal{B}$ measurable and gaussian is clear. Note that from (6.6) it follows that

$$I_{f_*} = \eta_{f_*} + \int_0^T \hat{\epsilon}_{C^*(t)f_*(t)}(t)dt \tag{6.38}$$

hence the property (6.36) is immediate. The proof of (6.37) is either direct, or a consequence of Ito's formula.
Indeed, let

$$\mathcal{B}^t = \sigma\Big(Z_{f_*(\cdot)1_{(0,t)}},\quad \forall f_* \in L^2(0,T;F')\Big)$$

then $\hat{\epsilon}_h(t)$ is independant from $\mathcal{B}^t$. Assume to simplify a little that (6.36) holds, then we can use Ito's formula to assert that

$$\begin{aligned} E\ I_{f_*^1}\ I_{f_*^2}\ &=\ E\ \int_0^T\ I_{f_*^2 1_{(0,t)}}\ \hat{\epsilon}_{C^*(t)f_*^1(t)} \\ &+E\int_0^T\ I_{f_*^1 1_{(0,T)}}\ \hat{\epsilon}_{C^*(t)f_*^2(t)} + \int_0^T < R(t)f_*^1(t), f_*^2(t) > dt \end{aligned} \tag{6.39}$$

and from the independance property the two first terms at the right hand side of (6.39) vanish, and the result (6.37) follows.

7 A CLASS OF NON LINEAR PROBLEMS

7.1 Setting of the problem

We denote by

$$H_1 = \left\{ h = \Sigma \lambda_i h_i \;\middle|\; \Sigma \lambda_i^2 \, \alpha_i < \infty \right\} \tag{7.1}$$

where h_i is an orthonormal asis of H, $\Sigma\alpha_i < \infty$, $\alpha_i \geq 0$. We now consider $g : H_1 \to H$, $B : H_1 \to \mathcal{L}(E; H)$ with

$$\left| g(h) - g(h') \right|_H \;\leq\; c|h - h'|_{H_1} \tag{7.2}$$

$$\left\| B(h) - B(h') \right\|_{\mathcal{L}(E;H)} \;\leq\; c|h - h'|_{H_1} \tag{7.3}$$

then we consider the following problem

$$y_h(t) + \int_0^t y_{A^*(\tau)h}(\tau)d\tau = \xi_h + \int_0^t (h, g(y(\tau)))d\tau + \xi_{B^*(y(0))h\mathbf{1}_{(0,t)}} \tag{7.4}$$

$$\forall h \in V, \quad \forall t, \quad \text{a.s.}$$

in which the notation $y(t)$ inside g and B^*, means

$$y(t) = \sum \alpha_i y_{h_i}(t) h_i \tag{7.5}$$

7.2 Statement of the result

We then have the

Theorem5.2 *Assume (7.2), (7.3) then there exists one and only one adapted continuous process* $(y(\cdot) \in L(\Omega, \mathcal{A}, P; L^\infty(0,T;H))$ *such that (7.4) holds. Moreover, recalling the definitions (5.5), (5.6) one has the relations*

$$y_h(t) = \xi_{q_{t,h}(0)} + \int_0^t \Big(q_{t,h}(\tau), g(y(\tau))\Big)d\tau + \xi_{B^*(y(\cdot))\, q_{t,h}(\cdot)\mathbf{1}_{(0,t)}} \tag{7.6}$$

Proof.

Assume first T not too large. let us consider a process $z(t,\omega)$ with values in H_1 , adapted to the filtration $\mathcal{E}^t$ and continuous. We define the L.R.F.

$$\eta_h(t) = \xi_{q_{0,h}(0)} + \int_0^t \Big(q_{t,h}(\tau), g(z(\tau))\Big)d\tau + \xi_{B^*(z(\cdot))q_{t,h}(\cdot)1_{(0,t)}} \tag{7.7}$$

and the process

$$\eta(t) = \Sigma\alpha_i\eta_{h_i}(t)h_i. \tag{7.8}$$

Note first that $\eta_h(t)$ is a continuous, adapted process (see Proposition 3.2); Moreover

$$E\sup_{0\leq t\leq T}\eta_h(t)^2 \leq C(1+T)|h|^2\Big(1+E\int_0^T (z(t))\Big|^2_{H_1}dt\Big)$$

and

$$E\sup_{0\leq t\leq T}|\eta(t)|_1^2 \leq C(1+T)\Big(1+E\int_0^T (z(t))\Big|^2_{H_1}dt\Big) \tag{7.9}$$

where we have used the martingale properties of $\xi_{B^*(z(\cdot))q_{t,h}(\cdot)1_{(0,t)}}$. Calling

$$P^m\eta(t) = \sum_{i=1}^m \alpha_i\eta_{h_i}(t)h_i \tag{7.10}$$

we have also

$$E\sup_{0\leq t\leq T}\Big|P^m\eta(t)-\eta(t)\Big|_1^2 \leq (\sum_{i=m+1}^{\infty}\alpha_i)c(1+T)(1+E\int_0^T|z(t)|^2_{H_1}dt) \tag{7.11}$$

which implies that $\eta(t)$ is an adapted continuous process with values in H_1. So, we have defined a map K_T from the closed subspace of $L^2(\Omega,\mathcal{A},P;L^\infty(0,T;H_1))$ made of adapted continuous processes, into itself.

Let us check that K_T is a contraction for T sufficiently small. Indeed, let z and $\tilde{z}$ and their images $\eta,\tilde{\eta}$

$$\eta(t)-\tilde{\eta}(t) = \sum_i\alpha_i\int_0^t(q_{t,h_i}(\tau),g(z(\tau))-g(\tilde{z}(\tau)))d\tau\ h_i$$

$$+\sum_i\alpha_i\ \xi_{(B^*(z(\cdot))-B^*(\tilde{z}(\cdot)))q_{t,h_i}(\cdot)1_{(0,t)}}h_i$$

and thus, thanks to assumptions (7.2), (7.3), one has

$$E \sup_{0\le t\le T} |\eta(t) - \tilde{\eta}(t)|^2_{H_1} \le C(T+T^2)E \sup_{0\le t\le T} |z(t) - \tilde{z}(t)|^2_{H_1}.$$

If $C(T+T^2) < 1$, we obtain the contraction property; The fixed point of K_T is the unique solution of (7.6). Since we can solve (7.6) in $(0,T)$, $(T,2T)$, we have obtained the desired result. From (7.6), we recover easily (7.4), proceeding as in section 5.3, with some obvious changes. □

Remark 7.1. Assumption (7.2), (7.3) are satisfied when

$$\begin{aligned} g(h) &= g((h,h_1),\cdot,(h,h_m)) \\ B(h) &= B((h,h_1),\cdot,(h,h_m)) \end{aligned}$$

with

$$|g(x_1,\cdots,x_m) - g(x_1',\cdots,x_m')|_H \le C\Big(\sum(x_i - x_i')^2\Big)^{1/2}$$

$$\|B(x_1,\cdots,x_m) - B(x_1',\cdots,x_m')\|_{\mathcal{L}(E;H)} \le C\Big(\sum_i(x_i - x_i')^2\Big)^{1/2}.$$

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