# Difference Equations on Weighted Graphs*

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## 1 Introduction

In discrete systems graphs represent a basic tool to study links between agents. There has been recently in the literature interesting articles whose goal is to mimic on graphs well known problems of PDE (Partial Differential Equations) type. In particular the discrete Laplacian has been considered and the discrete Green Function introduced and studied (see Chung and Yau [2]). More recently, Berenstein and Chung [1] went a step further in the analogy, by introducing discrete gradients, Dirichlet Principle, considering Dirichlet and Neumann problems and identification questions. Since we are dealing with finite dimensional problems, formulations are more or less equivalent and it is meaningful to look for the simplest presentation. We try here to mimic the variational formulation of boundary value problems, in the spirit of Lions [4]. It is an approach which leads to simpler presentations. As it will be seen, we prefer to work with un-normalized weights rather than normalized weights. There is no loss whatsoever, and in addition the normalization

*Dedicated to Dr. Claude LeMarechal for his 60th anniversary prevents a clear variational formulation of the problem to hold (although of course by the equivalence mentioned earlier one can always recover it). We proceed with giving the probabilistic interpretation of the solution of the discrete P.D.E. (in fact, these problems may arise from Markov Chains). On the other hand, we extend part of the results in Berenstein and Chung [1] to oriented graphs with non-symmetric weights.

# 2 Assumptions and Notation

We give now our main definitions, assumptions and notations.

## 2.1 Connected Undirected Graphs

We consider a graph $G$ made of $N$ nodes (or vertices). When there is a link (called an edge) between two nodes $x, y$, we say that the nodes are adjacent. We denote an edge by $\{x, y\}$ (as a set notation) so that we have $\{x, y\} = \{y, x\}$, which means that there is no order in the link, or that the graph is *undirected*. We use the notation $x \sim y$ for two adjacent nodes. Note that we may have a link $\{x, x\}$ or not. We say that $G$ is *connected* whenever, for any pair of nodes $x, y$ there is a (finite) sequence (called a path or chain) $x_0 = x,\ x_1, \ldots, x_{n-1},\ x_n = y$ such that $x_{i-1} \sim x_i$, for every $i = 1, \ldots, n$. In the first part, only connected undirected graphs are considered.

We shall next consider subgraphs $S \subset G$ and say that $S$ is *induced* from $G$, whenever for every $x$ and $y$ in $S$ any chain connecting $x$ and $y$ is exclusively made of nodes belonging to $S$. Clearly, this implies (but it not equivalent) that all edges from $G$ which connect nodes in $S$ are actually links in $S$. We introduce the *boundary* of $S$

$$\partial S = \{x \notin S \mid \exists y \in S, \ \text{ such that } \ y \sim x\}. \tag{2.1}$$

Note the following property

$$\text{if } \ x, x' \in \partial S, \ \text{ then } \ x \not\sim x', \tag{2.2}$$

which follows immediately from the definition of an induced graph. Indeed, let $x$ and $x'$ be in $\partial S$ then there exist $y$ and $y'$ in $S$ such that $x \sim y$ and $x' \sim y'$. Hence, if $x \sim x'$ then there is a path (or chain) joining $y$ and $y'$ two edges in $S$ and because the subgraph $S$ is induced, then the whole path must be in $S$, i.e., $x$ and $x'$ are in $S$, contradicting the definition of the boundary $\partial S$.

We shall also need the concept of *inner boundary*

$$\partial\dot{S} = \{x \in S \mid \exists\, y \in \partial S, \ \text{ such that } \ y \sim x\}.$$

We call $\bar{S} = S \cup \partial S$. Naturally a node in $S$ cannot be adjacent to a node in $G \setminus \bar{S}$.

## 2.2 Functions on Graphs

We shall consider (numerical) functions $f(x)$ defined for $x$ in $G$. These functions form a finite dimensional space $R^N$. By analogy with functional spaces we define

$$\int_G f = \sum_{x \in G} f(x) \tag{2.3}$$

and the (Hilbert) space $L^2(G)$ composed by all functions $f : G \to \mathbb{R}$, with the scalar product and the norm

$$(f, g) = \int_G fg, \qquad \|f\|^2_{L^2} = \int_G f^2. \tag{2.4}$$

We next define the partial (or directional) derivatives

$$\partial_y f(x) = f(y) - f(x) \tag{2.5}$$

and the gradient

$$Df(x) = \partial_y f(x), y \in G \tag{2.6}$$

Clearly $\partial_y f(x) = -\partial_x f(y)$. We introduce a semi-norm in $L^2(G)$ by setting

$$\|\!|f|\!\| = \sqrt{\sum_x \sum_{y \sim x} \left(\partial_y f(x)\right)^2},$$

and we have

**Lemma 2.1.** *The following property holds: if* $\|\!|f|\!\| = 0$ *then* $f$ *is constant.*

*Proof.* From the definition, if $[\![f]\!] = 0$ then $f(x) = f(y)$ for every $y \sim x$. But if $x, y$ are two arbitrary nodes then there exists a chain $x_0 = x$, $x_1, \dots, x_{n-1}$, $x_n = y$ such that $x_{i-1} \sim x_i$, for every $i = 1, \dots, n$. Therefore $f(x_{i-1}) = f(x_i)$, for every $i = 1, \dots, n$. It follows that $f(x) = f(y)$, for all $x, y$ in $G$. Hence the result. □

Based on the previous Lemma, we may define the subspace $H^1(G)$ of $L^2(G)$ composed by all functions with zero-average, thus, if the average is given by

$$\langle f \rangle = \frac{1}{N} \int_G f$$

then

$$H^1(G) = \big\{ f \in L^2(G) \ : \ \langle f \rangle = 0 \big\}, \tag{2.7}$$

with the norm defined by

$$\|f\|_{H^1} = [\![f]\!] = \sqrt{\sum_x \sum_{y \sim x} \big(\partial_y f(x)\big)^2}, \tag{2.8}$$

and the scalar product

$$(\!(f, g)\!) = \sum_x \sum_{y \sim x} \partial_y f(x) \partial_y g(x).$$

The notation is reminiscent of that of Sobolev spaces. Usually, for functions in $H^1(G)$ we denote the norm by $\|\cdot\|$ or explicitly $\|\cdot\|_{H^1}$. However, for any function in $L^2(G)$ we use the semi-norm notation $[\![\cdot]\!]$, as in (2.8).

We may re-interpret the subspace $H^1(G)$ as follows. First, define an equivalence relation on $L^2(G)$ by setting

$$f \sim g \ \text{ iff } \ f(x) - g(x) \ \text{ is a constant}$$

Next, consider the quotient space $\tilde{L^2}(G) = L^2(G)/\sim$, equipped with the norm

$$\|\tilde{f}\|_{\tilde{L^2}} = \|f - \langle f \rangle\|_{L^2}, \tag{2.9}$$

where $\tilde{f}$ is an equivalence class, $f$ a representative. Clearly, we may take as representative of $\tilde{f}$ the element with zero-average.

The $H^1$ semi-norm becomes a norm on the quotient space, which is equivalent to the quotient norm. In other words we have the inequalities

$$c_0 \|f - \langle f \rangle\|_{L^2} \leq \|f\|_{H^1} \leq c_1 \|f - \langle f \rangle\|_{L^2}, \tag{2.10}$$

for every function $f$ in $L^2(G)$ and some positive constants $c_1 \geq c_0 > 0$. Thus, the (sub-)space $H^1(G)$ is identified with the quotient space $L^2(G)/\sim$ equipped with the norm $H^1$, given by (2.8).

For any function $f(x, y)$ of two variables we may apply the directional derivative in the first variable, i.e., $\partial_y f(x, y) = f(y, y) - f(x, y)$. Therefore, based on the product formula

$$\begin{cases} \partial_y \big(f(x, y) g(x, y)\big) = \partial_y f(x, y)\, g(y, y) + f(x, y)\, \partial_y g(x, y) = \\ \qquad = \partial_y f(x, y)\, g(x, y) + f(y, y)\, \partial_y g(x, y) = \\ \qquad = \partial_y f(x, y)\, g(x, y) + f(x, y)\, \partial_y g(x, y) + \partial_y f(x, y)\, \partial_y g(x, y), \end{cases} \tag{2.11}$$

it is easy to check the following integration by parts formula

$$\begin{cases} \sum_x \sum_y \big[\partial_y f(x, y)\, g(x, y) + f(x, y)\, \partial_y g(x, y)\big] = \\ \qquad = \sum_x \sum_y \big[\partial_y \big(f(x, y) g(x, y)\big) - \partial_y f(x, y)\, \partial_y g(x, y)\big], \end{cases}$$

which reduces to

$$\sum_x \sum_y \big(\partial_y f(x, y)\big) g(x, y) = \sum_x \sum_y \partial_y \big(f(x, y) g(x, y)\big), \tag{2.12}$$

whenever $f(y, y) = 0$ for every $y$.

Let $S$ be an induced subgraph with its boundary $\partial S$. We consider now spaces of functions on $\bar{S}$. The only difference between $S$ and $G$ is the boundary, however, $\bar{S}$ and $G$ are similar. As above, define

$$\int_{\bar{S}} f, \quad \|f\|_{L^2(\bar{S})}, \quad L^2(\bar{S}), \quad H^1(\bar{S}) = L^2(\bar{S})/\sim, \quad H^1_0(S).$$

Note that a function $f \in L^2(\bar{S})$ can be considered as a function in $L^2(G)$ which vanishes outside $\bar{S}$, however, one cannot consider a function in $H^1(\bar{S})$ as a function in $H^1(G)$ which vanishes outside $\bar{S}$. The same difference exists in the usual Sobolev spaces. This is due to the contribution of edges between nodes of $\partial S$ and nodes outside $\bar{S}$. Let $H_0^1(S)$ be the space of functions on $G$ which vanish outside $S$ equipped with the $H^1$ semi-norm. As in Lemma 2.1, one can check that the semi-norm is a norm, and explicitly

$$\|f\|_{H_0^1(S)}^2 = \sum_{x\in\bar{S}} \sum_{y\in\bar{S},\, y\sim x} \big(\partial_y f(x)\big)^2. \tag{2.13}$$

In particular, we can indifferently consider $f$ in $\bar{S}$ or in $G$, extended outside $\bar{S}$ by 0. The analogy is complete with the usual Sobolev space $H_0^1$. Note that $H^1$ may be regarded as a quotient space (each class of equivalence represented by an element in $L^2$ with zero average) while $H_0^1(S)$, with a non empty boundary $\partial S$, is a subspace of $L^2(\bar{S})$.

# 3 Variational Problems

In most of what follows and with the previous notation, we consider an undirected connected graph $G$ and an induced (connected) subgraph $S$ of $G$.

## 3.1 Weights and Bilinear Forms

To the edges of $G$, we associate weights. More precisely we consider a function on $G\times G$ called $\varpi(x,y)$, with the properties

$$\begin{cases} \varpi(x,y) = \varpi(y,x) \geq 0 \\ \varpi(x,y) > 0 \iff x\sim y \end{cases} \tag{3.1}$$

We define on $H^1(G)\times H^1(G)$ the bilinear form

$$\begin{cases} a(u,v) = \dfrac{1}{2}\displaystyle\sum_x\sum_y \varpi(x,y)\big(u(y)-u(x)\big)\big(v(y)-v(x)\big) = \\ \qquad\qquad = \dfrac{1}{2}\displaystyle\int_G\int_G \varpi(x,y)\partial_y u(x)\partial_y v(x) \end{cases} \tag{3.2}$$

We begin with several remarks. First, concerning the symmetry assumption on $\varpi(x,y)$. If we leave this assumption, then we note that

$$\sum_x\sum_y \varpi(x,y)\big(u(y)-u(x)\big)\big(v(y)-v(x)\big) =$$
$$= \sum_x\sum_y \varpi(y,x)\big(u(y)-u(x)\big)\big(v(y)-v(x)\big) =$$

and therefore we obtain the same bilinear form with $\varpi(x,y)$ changed into $\frac{1}{2}(\varpi(y,x)+\varpi(x,y))$. Thus, the symmetry is not really an assumption. Second, we have $a(u,v)=a(v,u)$ for every $u$ and $v$ in $H^1(G)$, and thirdly, the properties (3.1) of $\varpi(x,y)$ yield the following estimate

$$\begin{cases} |a(u,v)| \leq M\|u\|_{H^1(G)}\,\|v\|_{H^1(G)}, \\ a(u,u) \geq \alpha\|u\|^2_{H^1(G)}, \end{cases} \tag{3.3}$$

for any $u$ and $v$ in $H^1(G)=\tilde{L^2}(G)$ and with $M$, $\alpha$ some positive constants, namely,

$$\begin{cases} M = \max\big\{\frac{1}{2}\varpi(x,y)\ :\ x,y\in G\big\}, \\ \alpha = \min\big\{\frac{1}{2}\varpi(x,y)\ :\ x,y\in G,\ \varpi(x,y)>0\big\}. \end{cases}$$

Finally, we state the property, which leads to the *Maximum Principle*

$$a(u,u^-)\leq 0, \qquad a(u,u^+)\geq 0 \quad \forall\, u,v\in H^1(G), \tag{3.4}$$

where $u^-(x)=-\min(0,u(x))$, and $u^+(x)=\max(0,u(x))$. Indeed, by checking all possibilities one deduce the elementary inequality

$$(a-b)(a^+-b^+)\geq 0, \quad \forall\, a,b\in\mathbb{R},$$

which yields the above property (3.4). Finally, it is clear that if we consider $a(u,v)$ as a bilinear form on $L^2(G)$, the coercivity does not hold anymore.

**Lemma 3.1.** *Under the assumption* (3.1), *define the operator*

$$Au(x) = -\sum_y \partial_y\big(\varpi(x,y)\partial_y u(x)\big). \tag{3.5}$$

*from $L^2(G)$ into itself. Then $\langle Au\rangle=0$ for every function $u$, so that $A$ can be considered as an operator from $H^1(G)$ into itself. Also, we have the property*

$$a(u,v)=(Au,v), \quad \forall\, u,v\in L^2(G). \tag{3.6}$$

*and $A$ is self-adjoint positive definite on $H^1(G)$.*

*Proof.* We first check easily that, one has also

$$Au(x) = \sum_y \varpi(x,y)(u(x) - u(y)) \tag{3.7}$$

and thus, by symmetry,

$$\sum_x \sum_y \varpi(x,y)(u(x) - u(y)) = 0,$$

i.e., $\langle Au \rangle = 0$. This shows that $Au$ is the representative (with zero-average) of an element of $\tilde{L}^2(G)$. Since

$$\sum_x \sum_y \varpi(x,y)\partial_y u(x)\partial_y v(x) = \sum_x Au(x)\, v(x) + \\ + \sum_x \sum_y \varpi(x,y)\partial_y u(x)\, v(y),$$

the symmetry properties $\varpi(x,y) = \varpi(y,x)$ and $\partial_y v(x) = -\partial_x v(y)$ yield

$$\sum_x \sum_y \varpi(x,y)\partial_y u(x)\, v(y) = -\sum_x \sum_y \varpi(y,x)\partial_x u(y)\, v(y).$$

Hence

$$\sum_x \sum_y \varpi(x,y)\partial_y u(x)\partial_y v(x) = 2\sum_x Au(x)\, v(x),$$

i.e.,

$$a(u,v) = \sum_x Au(x)v(x),$$

and property (3.7) holds. □

Let $S$ be an induced subgraph with its boundary $\partial S$. We consider the bilinear form on $H^1(\bar{S}) \times H^1(\bar{S})$

$$a_{\bar{S}}(u,v) = \frac{1}{2} \sum_{x \in \bar{S}} \sum_{y \in \bar{S}} \varpi(x,y)(u(y) - u(x))(v(y) - v(x)) \tag{3.8}$$

This form does not coincide with $a(u,v)$ for any functions $u$ and $v$ in $L^2(G)$ with $u = v = 0$ on $G \setminus \bar{S}$, since we need $u = v = 0$ on $G \setminus S$. Hence, one has the property

$$a(u,v) = a_{\bar{S}}(u,v), \quad \forall\, u, v \in H_0^1(S), \tag{3.9}$$

see (2.13).

Proceeding as in Lemma 3.1, we define the operator from $\tilde{L^2}(\bar{S})$ into itself such that

$$a_{\bar{S}}(u,v) = (A_{\bar{S}}u, v), \quad \forall\, u, v \in H_0^1(S), \tag{3.10}$$

with

$$A_{\bar{S}}u(x) = \sum_{\{y \in \bar{S}\}} \varpi(x,y)(u(x) - u(y)) \tag{3.11}$$

The operators $A$ and $A_{\bar{S}}$ do not coincide, otherwise we see easily that

$$A_{\bar{S}}u(x) = Au(x) \quad \forall\, x \in S,\ \forall\, u \in L^2(G) \tag{3.12}$$

For any $u$ in $L^2(\bar{S})$ we define the co-normal derivative relative to the operator $A$ as follows

$$\frac{\partial u}{\partial \nu_A}(x) = \sum_{\{y \in S\}} \varpi(x,y)(u(x) - u(y)), \quad \forall\, x \in \partial S. \tag{3.13}$$

To simplify notation, and unless otherwise mentioned, we may use $\partial \nu$ instead of $\partial \nu_A$ to symbolize the co-derivative relative to $A$. We can then state the Green's formula

**Lemma 3.2.** *Assume that $S$ is an induced subgraph of an undirected connected graph $G$ with a weight satisfying* (3.1). *Then we have the formula*

$$\int_S Au\, v + \int_{\partial S} \frac{\partial u}{\partial \nu}\, v = \int_S Av\, u + \int_{\partial S} \frac{\partial v}{\partial \nu}\, u, \tag{3.14}$$

*for every $u$ and $v$ in $L^2(\bar{S})$.*

*Proof.* Since $A_{\bar{S}}$ is self adjoint in in $L^2(\bar{S})$ we have

$$\int_{\bar{S}} A_{\bar{S}} u\, v = \int_{\bar{S}} A_{\bar{S}} v\, u.$$

This relation implies

$$\int_S A_{\bar{S}} u\, v + \int_{\partial S} A_{\bar{S}} u\, v = \int_S A_{\bar{S}} v\, u + \int_{\partial S} A_{\bar{S}} v\, u.$$

Now for $x$ in $\partial S$ one has

$$\begin{aligned} A_{\bar{S}} u(x) &= \sum_{\{y\in\bar{S}\}} \varpi(x,y)(u(x)-u(y)) = \\ &= \sum_{\{y\in S\}} \varpi(x,y)(u(x)-u(y)). \end{aligned}$$

Therefore

$$A_{\bar{S}} u(x) = \frac{\partial u}{\partial \nu}(x), \quad \forall\, x \in \partial S.$$

Taking also into account (3.12), the formula (3.14) follows immediately. □

Note that

$$A_{\bar{S}} u(x) = \frac{\partial u}{\partial \nu_A}(x), \quad \forall\, x \in \partial S,\ \forall\, u \in L^2(G). \tag{3.15}$$

## 3.2 Problem in the Full Graph

We first consider a problem posed in the full graph $G$. We consider a linear form on $H^1(G) = \tilde{L^2}(G)$, which can be represented as a linear form on $L^2(G)$, $(f,v)$ such that

$$(f,1) = \sum_y f(x) = 0. \tag{3.16}$$

We consider the equation

$$a(u,v) = (f,v), \quad \forall\, v \in H^1(G), \tag{3.17}$$

or, which is equivalent

$$Au = f \ \text{ in } \ G, \tag{3.18}$$

then we can state

**Theorem 3.3.** *Under the assumptions* (3.1) *and* (3.16) *there exists one and only one solution of* (3.17) *or* (3.18) *in* $H^1(G)$ *(or equivalently, a solution* $u$ *in* $L^2(G)$ *with* $\langle u \rangle = 0$*). Moreover one can represent the solution as*

$$u(x) = \int_G G(x,y) f(y). \tag{3.19}$$

*The Green function* $G(x,y)$ *is given by the formula*

$$G(x,y) = \sum_{j=1}^{N-1} \frac{1}{\lambda_j} \Psi_j(x) \Psi_j(y). \tag{3.20}$$

*where* $\lambda_j$ *and* $\Psi_j$ *are the eigenvalues and eigenfunctions of the operator* $A$*. We have*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}, \quad A\Psi_j(x) = \lambda_j \Psi_j(x).$$

*Proof.* Since $H^1(G)$ is a Hilbert space of dimension $N-1$ (finite dimensional) and the operator $A$ is self-adjoint, positive definite, there exist $N-1$ eigenvalues and eigenfunctions $\lambda_j$ and $\Psi_j$. The $\Psi_j$ form an orthonormal system in $\tilde{L^2}(G)$, namely

$$\int_G \Psi_j \Psi_k = \delta_{j,k}, \quad \langle \Psi_j \rangle = 0.$$

Because $\lambda_j = (A\Psi_j, \Psi_j)$, we deduce that $\lambda_j > 0$. Hence, the representation (3.19) follows immediately. □

We can compare this result with that of Berenstein and Chung [1]. These authors introduce the notation

$$d_\varpi(x) = \sum_y \varpi(x,y),$$

and $d$ is the operator

$$dv(x) = v(x) d_\varpi(x).$$

In the notation of Berenstein and Chung [1]

$$-\Delta_\varpi = d^{-1} A$$

namely,

$$-\Delta_\varpi u(x) = u(x) - \frac{\sum_y \varpi(x,y)u(y)}{d_\varpi(x)}.$$

They then solve the problem

$$-\Delta_\varpi u(x) = g(x).$$

This amounts to $Au(x) = dg(x)$, and the solvability condition becomes

$$\int_G g(x)d_\varpi(x) = 0.$$

So from Theorem 3.3, it follows that the solution $u$ is given by

$$u(x) = \sum_{j=1}^{N-1} \frac{1}{\lambda_j}\Psi_j(x)\sum_y \Psi_j(y)g(y)d_\varpi(y) \tag{3.21}$$

To cope with the fact that $-\Delta_\varpi$ is not self-adjoint, Berenstein and Chung [1] introduce

$$\mathcal{L}_\varpi = d^{\frac{1}{2}}(-\Delta_\varpi)d^{-\frac{1}{2}} = d^{-\frac{1}{2}}Ad^{-\frac{1}{2}},$$

which is also self-adjoint nonnegative. Setting $\tilde{u} = d^{\frac{1}{2}}u$ then $\tilde{u}$ is solution of $d^{-\frac{1}{2}}Ad^{-\frac{1}{2}}\tilde{u}(x) = d^{\frac{1}{2}}g(x)$. Considering the system of eigenvalues $\mu_j$ and eigenvectors $\Phi_j$ of the operator $d^{-\frac{1}{2}}Ad^{-\frac{1}{2}}$, and noting that $\mu_0 = 0$, $\mu_j > 0$, for every $j > 0$, one obtains

$$\tilde{u}(x) = c\sqrt{d_\varpi(x)} + \sum_{j=1}^{N-1}\frac{1}{\mu_j}\Phi_j(x)\sum_y \Phi_j(y)g(y)\sqrt{d_\varpi(y)},$$

where $c$ is an arbitrary constant. Thus

$$u(x) = c + \sum_{j=1}^{N-1}\frac{1}{\mu_j}\frac{\Phi_j(x)}{\sqrt{d_\varpi(x)}}\sum_y \Phi_j(y)g(y)\sqrt{d_\varpi(y)} \tag{3.22}$$

The formulas (3.21) and (3.22) are of course equivalent. But (3.21) is simpler and leads immediately to a choice of the constant so that $\langle u\rangle = 0$.

## 3.3 Dirichlet Problem

In this section, we consider an induced subgraph $S$ of an undirected connected graph $G$ with a weight $\varpi$. The boundary $\partial S$ is assumed to be nonempty, so that $S \neq G$. First, given a function $f : S \to \mathbb{R}$, we solve the Dirichlet problem: find a function $u$ such that

$$\begin{cases} Au(x) = f(x), \quad \forall\, x \in S, \\ u(x) = 0, \quad \forall\, x \in \partial S, \end{cases} \tag{3.23}$$

equivalently, this is the solution of the variational problem

$$\begin{cases} \text{Find } \; u \; \text{ in } \; H_0^1(S) \; \text{ such that} \\ a_{\bar{S}}(u, v) = (f, v), \quad \forall v \in H_0^1(S). \end{cases} \tag{3.24}$$

Recalling that $a(u, v) = a_{\bar{S}}(u, v)$ for any $u$ and $v$ such that at least $u$ or $v$ belongs to $H_0^1(S)$, i.e., vanishes on $G \setminus S$, we may replace $a_{\bar{S}}(u, v)$ by $a(u, v)$ in the above formulation of the variational problem. The space $H_0^1(S)$ has dimension $|S|$, the number of elements (nodes) of $S$. The operator $A$ is self-adjoint, positive definite. We can consider its system of eigenvalues and eigenvectors, $\mu_j$, $\Phi_j$, $j = 1, \ldots, |S|$. Therefore the solution of (3.23) is given by the formula

$$u(x) = \sum_{j=1}^{|S|} \frac{1}{\mu_j} \Phi_j(x) \sum_{\{y \in S\}} \Phi_j(y) f(y), \quad \forall\, x \in S, \tag{3.25}$$

i.e., in term of the Green function.

We turn now to the nonhomogeneous Dirichlet problem, that is to say

$$\begin{cases} Au(x) = 0, \quad \forall\, x \in S, \\ u(x) = g(x), \quad \forall\, x \in \partial S, \end{cases} \tag{3.26}$$

where $g : \partial S \to \mathbb{R}$ is a given function. We can reduce the nonhomogeneous Dirichlet problem to the homogeneous one, as follows. Let us define the operator

$$Bg(x) = \sum_{\{y \in \partial S\}} \varpi(x, y) g(y), \quad \forall\, x \in S. \tag{3.27}$$

Consider the homogeneous Dirichlet problem

$$\begin{cases} A\tilde{u}(x) = Bg(x), & \forall\, x \in S, \\ \tilde{u}(x) = 0, & \forall\, x \in \partial S. \end{cases} \tag{3.28}$$

We can state the

**Lemma 3.4.** *One has the property $u(x) = \tilde{u}(x)$, for every $x$ in $S$.*

*Proof.* Let us set $w = u - \tilde{u}$ then $w$ verifies

$$\begin{cases} Aw(x) = -Bg(x), & \forall\, x \in S, \\ w(x) = g(x), & \forall\, x \in \partial S. \end{cases}$$

Let us consider, for an arbitrary $f$ the solution of the Dirichlet problem

$$\begin{cases} Av(x) = f(x), & \forall\, x \in S, \\ v(x) = 0, & \forall\, x \in \partial S. \end{cases}$$

We then make use of Green's formula (3.14) to write

$$\int_S Aw\, v - \int_S Av\, w + \int_{\partial S} \frac{\partial w}{\partial \nu_A}\, v - \int_{\partial S} \frac{\partial v}{\partial \nu_A}\, w = 0.$$

Hence

$$-\int_S Bg\, v - \int_S f\, w - \int_{\partial S} \frac{\partial v}{\partial \nu_A}\, g = 0,$$

and it is easy to check that the first and third terms cancel. There remains

$$\int_S fw = 0.$$

Since $f$ is arbitrary, we deduce $w = 0$ in $S$. This completes the proof. □

We can state the following existence and uniqueness result

**Theorem 3.5.** *Let $S$ be a induced subgraph of an undirected connected graph $G$ with a weight satisfying* (3.1). *Assume that the boundary $\partial S$ is nonempty* *and consider two functions $f : S \to \mathbb{R}$ and $g : \partial S \to \mathbb{R}$. Then the Dirichlet problem*

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S, \\ u(x) = g(x), & \forall\, x \in \partial S, \end{cases} \tag{3.29}$$

*has one and only one solution, which is explicitly given using the Green function, namely,*

$$u(x) = \int_S G(x,y)\big[f(y) + Bg(y)\big], \quad \forall\, x \in S, \tag{3.30}$$

*where $Bg$ is defined by (3.27) and*

$$G(x,y) = \sum_{j=1}^{|S|} \frac{1}{\mu_j} \Phi_j(x)\, \Phi_j(y), \quad \forall\, x, y \in S, \tag{3.31}$$

*is the corresponding Green function. Moreover, the discrete maximum principle holds, i.e., if $f \geq 0$ and $g \geq 0$ then $u \geq 0$.* □

*Proof.* Since $g \geq 0$ implies $Bg \geq 0$, we need only to check the homogeneous case. Thus, from variational form (3.24) we deduce for $v = u^-$, which belongs to $H_0^1(S)$,

$$a(u, u^-) = (f, u^-) \geq 0,$$

because $f \geq 0$. On the other hand, by means of property (3.4) we get $a(u, u^-) = 0$. Hence

$$\big(u(y) - u(x)\big)\big(u^-(y) - u^-(x)\big) = 0 \ \text{ if } \ x \sim y.$$

Now, suppose we have a node $x_0$ in $S$ such that $u(x_0) < 0$, necessarily,

$$u(y) = u(x_0) < 0, \quad \forall\, y \sim x_0.$$

But if $z \in \partial S$, there is a path which connects $x_0$ to $z$. Necessarily $u(z) = u(x_0)$, but $u(z) = 0$, and we get a contradiction. □

Note that we have

$$u(x) = \int_S G(x,y) f(y) + \int_{\partial S} \frac{\partial G(x,y)}{\partial \nu_y}\, g(y), \quad \forall\, x \in S,$$

where the co-normal derivative is taken with respect to the second variable $y$ of the Green function. Thus, the maximum principle shows that $G(x,y) \geq 0$ and as well as its co-normal derivative in $y$.

## 3.4 Neumann Problem

We want to solve the problem

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S \\ \dfrac{\partial u}{\partial \nu}(x) = g(x), & \forall\, x \in \partial S \end{cases} \tag{3.32}$$

Considering the bilinear form $a_{\bar{S}}(u, v)$ and the operator $A_{\bar{S}}$ defined in (3.10), (3.11), we know from (3.12) and (3.15) that the problem is equivalent to

$$\begin{cases} A_{\bar{S}}u(x) = f(x), & \forall\, x \in S \\ A_{\bar{S}}u(x) = g(x), & \forall\, x \in \partial S \end{cases} \tag{3.33}$$

or, with the variational formulation

$$a_{\bar{S}}(u, v) = \ell_{\bar{S}}(v), \quad \forall\, v \in H^1(\bar{S}) \tag{3.34}$$

with

$$\ell_{\bar{S}}(v) = \int_S f\, v + \int_{\partial S} g\, v. \tag{3.35}$$

The solvability condition is given by

$$\ell_{\bar{S}}(1) = 0 \tag{3.36}$$

The solution of (3.35) is given in a similar way as for (3.17), with $\bar{S}$ playing the role of $G$ ($\bar{S}$ is considered as a graph with no boundary, the edges between the nodes of $\partial S$ and $G \setminus \bar{S}$ being suppressed). Details are left for the reader.

## 3.5 Non-symmetric Bilinear Forms

As long as we are dealing with an undirected graph $G$ the weight $\varpi$ must be symmetric, without going into details related to directed graphs (see Section 5.1 below), we may start with an operator of the form (3.7), i.e.,

$$Au(x) = \sum_y \varpi(x, y)\big(u(x) - u(y)\big), \quad \forall\, x \in G, \tag{3.37}$$

and define a bilinear form

$$a(u, v) = (Au, v), \quad \forall\, u, v \in L^2(G), \tag{3.38}$$

where now $\varpi(x,y) \geq 0$ for every $x$ and $y$ in $G$, not necessarily symmetric, but $\varpi(x,y) > 0$ if and only if there is a link from $x$ to $y$. Since $(Au, v) = (u, A^* v)$, with

$$A^* v(x) = \sum_y \varpi(y,x)\big(v(x) - v(y)\big) + v(x) \sum_y \big(\varpi(x,y) - \varpi(y,x)\big),$$

we deduce that

$$\begin{aligned} a(u,u) &= \frac{1}{2}\big((A + A^*)u, u\big) = \\ &= \frac{1}{2} \sum_{x,y} \big(\varpi(x,y) + \varpi(y,x)\big)\big(u(x) - u(y)\big)u(x) + \\ &\quad + \frac{1}{2} \sum_x \Big[ \sum_y \big(\varpi(x,y) - \varpi(y,x)\big)\Big] \big(u(x)\big)^2. \end{aligned}$$

If we define

$$\begin{cases} \lambda = -\min \big\{ \frac{1}{2} \sum_y \big(\varpi(x,y) - \varpi(y,x)\big) \ : \ x, y \in G \big\}, \\ \alpha = \min \big\{ \frac{1}{4}\big(\varpi(x,y) + \varpi(y,x)\big) \ : \ x, y \in G, \ \varpi(x,y) + \varpi(y,x) > 0 \big\} \end{cases}$$

then

$$a(u,u) + \lambda(u,u) \geq \alpha(\!(u,u)\!), \quad \forall\, u \in L^2(G),$$

meaning that the non-symmetric bilinear form $a(\cdot,\cdot)$ is coercive on $H^1(G)$ or $H^1_0(S)$ if $\lambda \leq 0$, i.e., if

$$\sum_y \big(\varpi(x,y) - \varpi(y,x)\big) \geq 0, \quad \forall\, x \in G. \tag{3.39}$$

Hence, under this condition (3.39) and as in the previous sections on symmetric bilinear forms, we can treat the case of the full graph $G$, and the Neumann and Dirichlet problems in an induced connected subgraph. Details are left to the reader. As we will see in Section 5 below, the condition (3.39) is not really necessary if we look at a non-variational formulation.

## 4 Identification

Our variational approach fits quite well with the problem of identification as considered by Berenstein and Chung [1]. For instance, the solution of the nonhomogeneous Dirichlet problem

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S, \\ u(x) = g(x), & \forall\, x \in \partial S \end{cases} \tag{4.1}$$

is the unique solution of the following minimization problem:
Minimize

$$\begin{cases} J(v) = a_{\bar{S}}(v,v) - 2\displaystyle\int_S fv, \quad \text{over the set} \\ \big\{v \in L^2(\bar{S}) \ : \ \text{ such that } v(x) = g(x), \forall x \in \partial S\big\}, \end{cases} \tag{4.2}$$

where $f$ and $g$ are the given functions.

Now, consider then two weights $\varpi_1(x,y), \varpi_2(x,y)$, such that

$$\varpi_1(x,y) \le \varpi_2(x,y), \quad \forall\, x,y \in \bar{S} \tag{4.3}$$

and denote by $A_1, A_2$ the operators corresponding to these weights. Let $u_1, u_2$ be two functions on $\bar{S}$, such that

$$\begin{cases} A_1u_1(x) = A_2u_2(x) = 0, & \forall\, x \in S \\ u_1(x) = u_2(x), & \forall\, x \in \partial S \\ \dfrac{\partial u_1}{\partial \nu_{A_1}}(x) = \dfrac{\partial u_2}{\partial \nu_{A_2}}(x), & \forall\, x \in \partial S, \end{cases} \tag{4.4}$$

then we state the

**Theorem 4.1.** *We assume* (4.3), (4.4)*, where $S$ is an induced subgraph of an undirected connected graph $G$ with a nonempty boundary $\partial S$. Then one has*

$$u_1(x) = u_2(x), \quad \forall\, x \in \bar{S} \tag{4.5}$$

*Moreover, if $u(x)$ denotes the common value of $u_1(x)$ and $u_2(x)$ then we have*

$$\varpi_1(x,y) = \varpi_2(x,y), \quad \forall\, x,y \in \bar{S} \text{ such that } u(x) \ne u(y) \tag{4.6}$$

*Proof.* Denote by $J_1(v)$ and $J_2(v)$ the functional (4.2) corresponding to the weights $\varpi_1$ and $\varpi_2$, respectively. Note that, since $f = 0$, one has

$$J_1(v) = a_{\{\bar{S},1\}}(v,v) = \frac{1}{2}\sum_{x\in\bar{S}}\sum_{y\in\bar{S}}\varpi_1(x,y)(v(y)-v(x))^2$$

and

$$J_2(v) = a_{\{\bar{S},2\}}(v,v) = \frac{1}{2}\sum_{x\in\bar{S}}\sum_{y\in\bar{S}}\varpi_2(x,y)(v(y)-v(x))^2.$$

Since

$$J_1(v) = \int_S A_1 v\, v + \int_{\partial S}\frac{\partial v}{\partial \nu_{A_1}}v,$$

assumption (4.4) yields

$$J_1(u_1) = J_2(u_2).$$

Now from (4.3), one has

$$J_2(u_2) \geq J_1(u_2)$$

therefore, necessarily (4.5) holds. We also have

$$\sum_{x\in\bar{S}}\sum_{y\in\bar{S}}(\varpi_1(x,y)-\varpi_2(x,y))(u(y)-u(x))^2 = 0$$

and, since all terms are negative, they all vanish, which implies (4.6). □

*Remark* 4.2. In Berenstein and Chung [1], one assumes in addition

$$\varpi_1(x,y) = \varpi_2(x,y), \quad \forall\, x\in\partial S,\ \forall\, y\in\partial\dot{S}. \tag{4.7}$$

In fact, this due to the difference of definition of the normal derivative. They call

$$\frac{\partial u}{\partial_\varpi n}(x) = \sum_{\{y\in S\}}\frac{u(x)-u(y)}{d'_\varpi x}$$

with

$$d'_{\varpi}x = \sum_{\{y \in S\}} \varpi(x, y), \quad \forall\, x \in \partial S.$$

In fact, the additional assumption (4.7) implies that

$$d'_{\varpi_1}x = d'_{\varpi_2}x, \quad \forall\, x \in \partial S.$$

Therefore the set of assumptions coincide. Our presentation is more synthetic. □

*Remark* 4.3. The fact that, in the first assumption (4.4), the right hand side is zero, is essential. The result fails even for an identical right hand side. □

*Remark* 4.4. If the additional assumption (4.7) of Remark 4.2 is made, and if in addition, the common value on the boundary is strictly positive, and

$$\int_S u_1(x) d_{\varpi_1}x = \int_S u_2(x) d_{\varpi_2}x$$

then

$$\varpi_1(x, y) = \varpi_2(x, y).$$

Indeed, we have $u_1(x) = u_2(x) = u(x)$ for every $x$ in $S$ and moreover $u(x) > 0$ for every $x$ in $S$. The desired result follows, see Berenstein and Chung [1], Theorem 4.1. □

# 5 Non-variational Formulation

In this section we outline the non-variational formulation of the problems mentioned in Section 3. We will be able to consider directed graphs $G$ which are not necessarily connected and subgraphs $S$ which are not necessarily induced. It is worth to remark that in the variational case, the existence of solutions to the Dirichlet and Neumann problems are consequences of the coercitivity condition on the bilinear form. Now, we will see that in the non-variational formulation, everything follows form the maximum principle.

## 5.1 Directed Graphs

As mentioned early, a (simple) *graph* is composed by a finite number of *vertices* (or nodes) $G$ with a subset $E$ of $G \times G$, whose elements are called *edges* (or arcs or links). Thus, a vertex $x$ is an element of $G$ and an edge connects two vertices. An edge of the form $(x, x)$ is possible and if $(x, y)$ is an edge then we say that the vertex $x$ is *adjacent* to $y$. A graph is called *undirected* if the set of edges is undirected, i.e., if an edge $(x, y)$ belongs to $E$ then $(y, x)$ also belongs to $E$ and $(x, y)$ is considered to be equal to $(y, x)$. For an undirected graph the edges are denotes by $\{x, y\}$ as a two-element subset of $G$ instead of the ordered pair $(x, y)$. A graph is *connected* if for every pair of vertices $x$ and $y$ there exits a (finite) sequence (termed a path or chain) of vertices $x = x_0, \ldots, x_n = y$ such that $(x_{i-1}, x_i)$ belongs to $E$, for any $i = 1, \ldots, n$, i.e., $x_{i-1}$ and $x_i$ are adjacent. A graph $(G', E')$ is called *subgraph* of $(G, E)$ or equivalent $(G, E)$ is called a *host graph* of $(G', E')$ if $G' \subset G$ and $E' \subset E$. A subgraph $(G', E')$ of $(G, E)$ is called an *induced* subgraph if every path (or chain) in $E$ connecting two nodes $x$ and $y$ in $E'$ is made entirely of edges in $E'$. It is clear that any induced subgraph of a connected graph is also connected.

A *weighted graph* $(G, E, \varpi)$ is a graph $(G, E)$ with a (weight) function $\varpi$ from $G \times G$ into $[0, \infty)$ satisfying $\varpi(x, y) > 0$ if and only if $(x, y)$ belongs to $E$. If the graph is undirected then $\varpi(x, y) = \varpi(y, x)$ is also required for every edge $\{x, y\}$. The *standard* weight function $\varpi$ takes only values 0 or 1 indicating the edges. The weight $\varpi(x, y)$ could represent the *capacity* (or conductivity) of the edge $(x, y)$ or $\{x, y\}$. Since the weight function $\varpi$ include all (and more) information relevant to determine the edges of a graph, a weighted graph is denoted by the couple $(G, \varpi)$ and the set of edges $E$ is defined as the pairs $(x, y)$ such that $\varpi(x, y) > 0$.

As in Section 2.2, and now for a directed graph $G$, we can define

$$\int_G f, \quad \|f\|_{L^2(G)}, \quad L^2(G).$$

Note that now edges are directed, i.e., the notation *adjacent* $\sim$ is not symmetric, $x \sim y$ means $\varpi(x, y) > 0$ which may not be the same as $y \sim x$, which means $\varpi(y, x) > 0$.

Given a directed weighted graph $(G, \varpi)$, and a function $\alpha : G \to \mathbb{R}$, define the operator

$$Au(x) = -\alpha(x)u(x) + \sum_y \varpi(x,y)\big(u(x) - u(y)\big), \tag{5.1}$$

for any function $u$ in $L^2(G)$. Recalling the inner product in $L^2(G)$ given by (2.4), we have

$$(Au, v) = (u, A^*v), \quad \forall\, u,\, v \in L^2(G), \tag{5.2}$$

where

$$\begin{cases} A^*v(x) = -\alpha^*(x)v(x) + \sum_y \varpi(y,x)\big(v(x) - v(y)\big), \\ \alpha^*(x) = \alpha(x) + \sum_y \big(\varpi(x,y) - \varpi(y,x)\big). \end{cases} \tag{5.3}$$

Note that

$$Au(x) = -\alpha(x)u(x) - \sum_y \varpi(x,y)\partial_y u(x)$$

and the *adjoint* operator $A^*$ is an operator similar to $A$, but corresponding to the (adjoint) weight $\varpi^*(x,y) = \varpi(y,x)$. Compare the discussion in Section 3.1 relative to a bilinear form, where $\varpi$ is necessarily symmetric. Even when $\alpha = 0$ we may have $\alpha^* \neq 0$ when the weight is not symmetric.

Consider a weighted graph $(G, \varpi)$, the *boundary* $\partial S$ and the *adjoint boundary* $\partial^* S$ of a subset $S$ of $G$ are defined by

$$\partial S := \{y \in G \setminus S \;:\; w(x,y) > 0 \text{ for some } x \in S\},$$
$$\partial^* S := \{y \in G \setminus S \;:\; w(y,x) > 0 \text{ for some } x \in S\}.$$

Clearly, referring $S$ as the inside and $G \smallsetminus S$ as the outside we see that $\partial S$ are outside nodes *reachable* from the inside, while $\partial^* S$ are outside nodes *reaching* the inside, both *communicating* (in opposite directions) the outside and the inside. It is clear that these two boundaries can be defined independently of the weight function, i.e., $\partial S$ and $\partial^* S$ depends on the (directed) edges of the graph.

If $S$ is an induced subgraph then the equivalent of property (2.2) can be phased as follows

$$\text{if } \; x \in \partial^* S \text{ and } y \in \partial S \; \text{ then } \; \varpi(x,y) = 0. \tag{5.4}$$

Indeed, if $x$ belongs to $\partial^* S$ then there exists a $x'$ in $S$ such that $\varpi(x, x') > 0$, and similarly, if $y$ to $\partial S$ then there exists a $y'$ in $S$ such that $\varpi(y', y) > 0$. Hence, if $\varpi(x, y) > 0$ then there is a path, $(y', y, x, x')$, joining the nodes $y'$ and $x'$, both in $S$. Because the subgraph is induced, the whole path should be in $S$, i.e., $x$ and $y$ must be in $S$, contradicting the definition of the boundaries $\partial S$ and $\partial^* S$.

Note that for any subset $S \subset G$ and any function $u : S \cup \partial S \to \mathbb{R}$, the above expression (5.1) defines a function $Au : S \to \mathbb{R}$, since $\varpi(x, y) = 0$ for every $x$ in $S$ and $y$ in $G \smallsetminus (S \cup \partial S)$. Similarly, for any function $v : S \cup \partial^* S \to \mathbb{R}$ we can define the function $A^* v : S \to \mathbb{R}$. We may extend the definition of $Au$ on the boundary $\partial^* S$ and $A^* v$ on boundary $\partial S$ as follows:

$$\begin{cases} \partial_\nu u(x) = -\alpha(x) u(x) + \displaystyle\sum_{y \in S} \varpi(x, y)\big(u(x) - u(y)\big), & \forall\, x \in \partial^* S, \\ \partial_\nu^* v(x) = -\alpha^*(x) v(x) + \displaystyle\sum_{y \in S} \varpi(y, x)\big(v(x) - v(y)\big), & \forall\, x \in \partial S. \end{cases} \tag{5.5}$$

Then, from relation (5.2) with $u(x) = 0$ for every $x$ in $G \smallsetminus (S \cup \partial S)$ and $v(x) = 0$ for every $x$ in $G \smallsetminus (S \cup \partial^* S)$ we deduce the integration by part formula

$$\int_S Au\, v + \int_{\partial^* S} \partial_\nu u\, v = \int_S u\, A^* v + \int_{\partial S} u\, \partial_\nu^* v, \tag{5.6}$$

for every $u$ in $L^2(S \cup \partial^* S)$ and $v$ in $L^2(S \cup \partial S)$. Clearly, $\partial_\nu$ and $\partial_\nu^*$ are the co-normal derivatives.

Thus, if we assume that

$$\partial S = \partial^* S \tag{5.7}$$

then the closure of $S$ is defined as

$$\bar{S} = S \cup \partial S = S \cup \partial^* S,$$

which is connected if $S$ is induced and the host graph $G$ is connected. Again, we may consider

$$\int_{\bar{S}} f, \quad \|f\|_{L^2(\bar{S})}, \quad L^2(\bar{S}),$$

as in Section 2.2, and clearly $Au(x)$ and $A^*v(x)$ are defined for any $u$ and $v$ in $L^2(\bar{S})$ and any $x$ in $S$. The spaces $H^1(G)$ and $H^1(\bar{S})$ are redefined by changing the normalization condition, see (5.10) below. The space $H_0^1(S)$ remains the same, however, the $H^1$-norm plays no role since the associated bilinear form is not used.

## 5.2 Discrete Maximum Principle

We have the following result

**Theorem 5.1 (Maximum Principle).** *Let $(G, \varpi)$ be a directed weighted graph, $\alpha$ be a real function defined on $G$, $S$ be a (non empty) subset of $G$ satisfying (5.7) such that its closure $\bar{S} = S \cup \partial S$ is connected* [1] *Suppose $u$ is a function from the closure $\bar{S}$ into $\mathbb{R}$ which attaints its global maximum at a node $x_0$ satisfying $\alpha(x_0)\, u(x_0) \geq 0$. Then*

$$Au(x_0) \leq 0 \;\; \text{if} \;\; x_0 \in S \quad \text{and} \quad \partial_\nu u(x_0) \leq 0 \;\; \text{if} \;\; x_0 \in \partial S,$$

*Moreover, if $Au(x) \geq 0$ for every $x$ in $S$ and $x_0$ belongs to $S$ then $u$ is a constant function and $\alpha(x_0)\, u(x_0) = 0$. Similarly, if $\partial_\nu u(x) \geq 0$ for every $x$ in $\partial S$ and $x_0$ belongs to $\partial S$ then $u$ is a constant function and $\alpha(x_0)\, u(x_0) = 0$.*

*Proof.* The first part follows from positivity condition on the weight, namely $\varpi(x, y) \geq 0$, and the definitions of the operator $A$ and the co-normal derivative $\partial_\nu$.

To check the second part, suppose that $Au(x) \geq 0$ for every $x$ in $S$. Since $u$ has its *global* maximum at $x_0$ we have

$$u(x_0) \geq u(x), \quad \forall x \in \bar{S}.$$

If $x_0$ belongs to $S$ then $\alpha(x_0)\, u(x_0) = 0$ and

$$\sum_y \big[u(y) - u(x_0)\big]\, \varpi(x_0, y) = 0,$$

where each term is nonnegative, so $u(y) = u(x_0)$ for every $y$ in $\bar{S}$ such that $\varpi(x_0, y) > 0$. Since $\bar{S}$ is connected we conclude that $u$ is a constant function in $\bar{S}$.

[1] If $S$ is induced then the closure $\bar{S}$ is connected whenever the host graph $G$ is so.

Similarly, we will show that if $x_0$ belongs to the boundary $\partial S$ then the co-normal derivative $\partial_\nu u(x_0) > 0$ unless $u$ is constant. Indeed, if $x_0$ belongs to the boundary $\partial S$ and the co-normal derivative vanishes then

$$\sum_{y\in S} \big[u(x_0) - u(y)\big]\varpi(y, x_0) = 0,$$

where all terms are nonnegative. Therefore, the definition of the boundary $\partial S$ show that at least for some $y_0$ in $S$ we must have $\varpi(y_0, x_0) > 0$ and so $u(x_0) = u(y_0)$. Hence, $u$ has a global maximum at $y_0$ in $S$ and the previous argument proves that $u$ must be a constant function on $\bar{S}$. □

*Remark* 5.2. The assumption that $\bar{S}$ is connected and the condition (5.7) are needed to simplify the statement of the discrete maximum principle. If $S$ is any subset of $G$, $u$ is a function defined on $S \cup \partial S \cup \partial^* S$ which attains its global maximum at $x_0$ satisfying $\alpha(x_0)\, u(x_0) \geq 0$ then

$$\begin{cases} Au(x_0) \leq 0 \ \text{ if } \ x_0 \in S, \\ \partial_\nu u(x_0) \leq 0 \ \text{ if } \ x_0 \in \partial^* S. \end{cases}$$

Moreover, denote by $S(x_0)$ the set of all nodes $y$ in $S \cup \partial S \cup \partial^* S$ having a path originated at $x_0$ and ending at $y$, and suppose that $Au(x) \geq 0$ for every $x$ in $S$. The following property holds: (1) if $x_0$ belongs to $S$ then $u$ is a constant function on $S(x_0)$, (2) if $x_0$ belongs to $\partial^* S$ then $\partial_\nu u(x_0) > 0$ unless $u$ is a constant function on $S(x_0)$. If also $\alpha^*(x_0)\, u(x_0) \geq 0$ and $x_0$ belongs to $\partial S$ then $\partial_\nu^* u(x_0) > 0$ unless $u$ is a constant function on $S(x_0)$. Another way of re-phasing this property is a follows: (1) if $Au(x) \geq 0$ for every $x$ in $S$, and $x_0$ belongs to $S$ and $\alpha(x_0)\, u(x_0) \geq 0$ then $u$ is a constant function on $S(x_0)$, (2) if $\partial_\nu u(x) \geq 0$ for every $x$ in $\partial S$, and $x_0$ belongs to $\partial S$ and $\alpha(x_0)\, u(x_0) \geq 0$ then $u$ is a constant function on $S(x_0)$. A typical application of this discrete maximum principle is when we assume the maximum value $u(x_0)$ nonnegative, as well as the functions $\alpha$ or $\alpha^*$ nonnegative. □

## 5.3 Problem without Boundary

Let $(G, \varpi)$ a directed weighted graph and $\alpha$ be a nonnegative function from $G$ into $\mathbb{R}$. Consider the operators $A$ and $A^*$ in $L^2(G)$, given by (5.1) and (5.3). Given a function $f : G \to \mathbb{R}$ we want to study the following problem: Find a function $u$ in $L^2(G)$ such that

$$Au(x) = f(x), \quad \forall\, x \in G, \tag{5.8}$$

which is related with its adjoint null problem: Find a function $m$ in $L^2$ such that

$$A^* m(x) = 0, \quad \forall\, x \in G. \tag{5.9}$$

Because we are in a finite dimensional space, the nullity of $A$ is equal to the nullity of $A^*$. If $h$ satisfies $Ah = 0$ then the discrete maximum principle implies that $h$ is constant, so that the nullity (i.e., the dimension of the null space or kernel) of $A$ is equal to 1 (the interesting case including when $\alpha = 0$) or 0 (including the case when $\alpha$ is strictly positive). Thus, the adjoint problem (5.9) do have a non-zero (if the nullity is equal to 1) solution $m$ (not necessarily constant), which can be normalized to satisfy

$$\int_G m = 1.$$

Hence, problem (5.8) do have a unique solution if and only if

$$\int_G f\, m = 0 = (f, m).$$

At this point and if the nullity is equal to 1, we modify the definition of the space $H^1(G)$ and the quotient space $\tilde{L}^2(G)$ to include the average with respect to $m$, the normalized solution of the adjoint problem (5.9), say, the *invariant measure*. Set

$$H^1(G) = \{ f \in L^2(G) \ : \ (f, m) = 0 \}, \tag{5.10}$$

which is the orthogonal complement of the kernel (or null space) of $A^*$. Clearly $A$ will map $H^1(G)$ into itself. Again, the discrete maximum principle ensures that $A$ is injective, so that $A$ is an invertible operator. This inverse $A^{-1}$ is representable as an *integral* operator called the Green function, i.e.,

$$u(x) = \sum_y G(x, y)\, f(y), \quad \forall\, x \in G. \tag{5.11}$$

is the unique solution of (5.8) satisfying $(u, m) = 0$ and under the assumption that $(f, m)$. Clearly, if the nullity is equal to 0 then $A$ is invertible in $L^2(G)$.

Now, let $S$ be a subset of $G$ where the boundary satisfies the condition (5.7). Then, $\bar{S}$ is playing the role of $G$, i.e., $\bar{S}$ is considered as a graph with no boundary, the edges between the nodes of $\partial S = \partial^* S$ and $G \smallsetminus \bar{S}$ being deleted. The space $H^1(\bar{S})$ is modified accordingly to the invariant measure. Results similar to the case $S = G$ hold true as long as the discrete maximum principle can be applied. Note that even when $\alpha$ changes sign, we may have $\alpha^*$ nonnegative, so that the nullity of $A^*$ may be calculated, and the same conclusions hold true.

Hence, we have proven the following results:

**Theorem 5.3.** *Let $S$ be a connected subgraph of a directed weighted host graph $(G, \varpi)$, and let $\alpha = 0$ in the expression (5.1) defining the operator $A$. Assume $\partial S = \partial^* S$ and set $\bar{S} = S \cup \partial S$. Given a function $f$ from $\bar{S}$ into $\mathbb{R}$, the discrete Neumann boundary problem*

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S, \\ \partial_\nu u(x) = f(x), & \forall\, x \in \partial S \end{cases} \tag{5.12}$$

*has a solution $u$, unique up to an additive constant, if and only if $(f, m) = 0$, where $m$ is the unique solution of the adjoint problem*

$$\begin{cases} A^* m(x) = 0, & \forall\, x \in S, \\ \partial_\nu^* m(x) = 0, & \forall\, x \in \partial S, \end{cases} \tag{5.13}$$

*which satisfies*

$$\int_{\bar{S}} m = 1. \quad \square \tag{5.14}$$

Note that in the above result, we may take $S = G$ so that $\partial S = \emptyset$. As mentioned above, if $\alpha \geq 0$ then the nullity cannot be 1 (so it is 0) and the solution $m$ of (5.13) can not be normalized because $m = 0$. In this case the problem (5.12) has a unique solution $u$ for every given $f$, i.e., no compatibility condition is needed.

On the other hand, starting from the definition (5.1) and (5.3) of the operators $A$ and its adjoint operator $A^*$, we may define a non-symmetric bilinear form by the expression

$$a(u, v) = (Au, v) = (u, A^* v), \quad \forall\, u, v \in L^2(G). \tag{5.15}$$

When $\alpha$ changes sign (i.e., it is not necessarily nonnegative), this bilinear form may be coercive in some subspace of $L^2(G)$ and the existence and uniqueness can be deduced. In this context, the bilinear form $a(\cdot,\cdot)$ is coercive in a subspace $V$ of $L^2(G)$ if (1) $a(u,v) \geq 0$ for every $u$ and $v$ in $V$ and (2) $a(u,u) = 0$ with $u$ in $V$ implies $u = 0$. For instance, if

$$\alpha(x) + \frac{1}{2}\sum_y \big(\varpi(x,y) - \varpi(y,x)\big) \geq 0, \quad \forall\, x \in G \tag{5.16}$$

then the non-symmetric form (5.15) is coercive in $H^1(G)$.

## 5.4 Dirichlet Boundary

Let $(G,\varpi)$ be a directed weighted graph and $\alpha$ be a nonnegative function from $G$ into $\mathbb{R}$. Consider the operators $A$ in $L^2(G)$, given by (5.1). Now, suppose given a proper and connected subset $S$ of $G$, and a function $f : G \to \mathbb{R}$, we want to study the following problem: Find a function $u$ in $L^2(G)$ such that

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S, \\ u(x) = f(x), & \forall\, x \in G \smallsetminus S. \end{cases} \tag{5.17}$$

Certainly, if the boundary $\partial S = \partial^* S$ is defined then the only relevant nodes are in the closure $\bar{S}$, no need to use the complement $G \smallsetminus S$. Note that the data $f$ may be considered as two functions, one defined on $S$ and another defined on $G \smallsetminus S$, however, we keep the notation $f$ for simplicity.

The same argument as before, the discrete maximum principle shows that the map $f \mapsto u$ is injective, from $L^2(G)$ into itself. Because we are working in a finite dimensional space, the above Dirichlet problem has one and only one solution $u$ which can be expressed as

$$u(x) = \sum_{y\in G} G(x,y)\, f(y), \quad \forall\, x \in G,$$

where $G(x,y)$ is the Green function. Clearly, the proper part of the Green function is $G(x,y)$ for $x$ and $y$ in $S$.

Note that as in Section 3.3, given a function $g$ defined on $G \smallsetminus S$, the expression

$$Bg(x) = \sum_{y\in G\smallsetminus S} \varpi(x,y) g(y), \quad \forall\, x \in S$$

transforms the following non-homogeneous (discrete) Dirichlet boundary value problem

$$\begin{cases} Au(x) = 0, & \forall\, x \in S, \\ u(x) = g(x), & \forall\, x \in G \setminus S \end{cases} \tag{5.18}$$

into a homogeneous problem

$$\begin{cases} Av(x) = Bg(x), & \forall\, x \in S, \\ v(x) = 0, & \forall\, x \in G \setminus S, \end{cases} \tag{5.19}$$

i.e., $u(x) = v(x)$ for every $x$ in $S$.

Even more general, let $S$ be a proper (non necessary connected) subset of $G$ and define the boundary $\Gamma = \partial S \cup \partial^* S$ and the closure $\overline{S} = S \cup \Gamma$. Suppose given a part of the boundary $\Gamma_0$ satisfying

$$\partial S \setminus \partial^* S \subset \Gamma_0 \subset \Gamma, \qquad \Gamma_0 \neq \emptyset, \tag{5.20}$$

and a function $f : \overline{S} \to \mathbb{R}$. Consider the mixed boundary problem

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S, \\ \partial_\nu u(x) = f(x), & \forall\, x \in \Gamma \setminus \Gamma_0, \\ u(x) = f(x), & \forall\, x \in \Gamma_0. \end{cases} \tag{5.21}$$

Now assume that $\Gamma_0$ is reachable form $S$, i.e., for every $x$ in $S$ there exists a path originated at $x_0 = x$ and ending at some point $y$ in $\Gamma_0$. Then we can use the discrete maximum principle, as stated in Remark 5.2, to check that the mixed problem (5.21) has a unique solution. Hence, the solution exits, it is unique and given by means of a Green function. In general, let $S_0$ be the component reaching $\Gamma_0$, i.e., the set of all nodes $x$ in $S$ having a path originated at $x_0 = x$ and ending at some point $y$ in $\Gamma_0$. Also, suppose that $S_k$, for $k = 1, \ldots, K$, $0 \leq K < |S|$, are the one-way closed and connected components of $S \setminus S_0$, i.e., $S$ is a disjoint union of $S_0$, $S_1$, ..., $S_K$ and each $S_k$ with $k \geq 1$ is connected, i.e., any two nodes in $S_k$ can be jointed with a path, and *one-way closed*, i.e., any path originated at a node in $S_k$ must have all its nodes also in $S_k$. Clearly, for every $k$ we can set independent problems. For $S_0$ we do have a problem similar to (5.21), for which existence and uniqueness hold true. For $S_k$ with $1 \leq k \leq K$ we have a Neumann boundary problem like the one in the previous section, namely,

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S_k, \\ \partial_\nu u(x) = f(x), & \forall\, x \in \partial^* S_k, \end{cases} \tag{5.22}$$

where $S_k$ is connected and $\partial S_k = \emptyset$. Because $\alpha(x) \geq 0$, the discrete maximum principle, as state in Remark 5.2, shows that the nullity of $A$ in $S_k$ is equal to 1 (e.g., if $\alpha(x) = 0$ for every $x$ in $S_k$) or is equal to 0 (e.g., if $\alpha(x) > 0$ for every $x$ in $S_k$). Then, for those $k$ where the nullity is 1 we find a normalized invariant measure $m_k$, solution of the adjoint problem in $S_k$. Therefore, the problem in the whole $S$ have a solution (unique after normalization) if and only if $f$ has a zero-average with respect to $m_k$ on $S_k$ for every $k$. Note that assumption (5.20) is essential to apply the discrete maximum principle, i.e., if $u$ is a solution of $Au = 0$ assuming its (global) maximum at $x_0$ in $\overline{S}$ then no conclusive statement can be deduced when $x_0$ belongs to $\partial S \smallsetminus \partial^* S$, since $Au(x)$ (or $\partial_\nu u(x)$) is only defined for $x$ in $S \cup \partial^* S$.

Again, as mentioned in the previous section, we may transform this mixed problem into a pure Dirichlet problem like (5.17), just delete all links in $G \smallsetminus \overline{S}$ and replace the full graph $G$ by $\overline{S}$ and $S$ by $\overline{S} \smallsetminus \Gamma_0$. Thus, if $A_S$ denotes the new operator in $\overline{S}$, which agree with $A$ in $S$ and with $\partial_\nu$ on $\Gamma \smallsetminus \Gamma_0$, then problem (5.21) is equivalent to

$$\begin{cases} A_S u(x) = f(x), & \forall\, x \in \overline{S} \smallsetminus \Gamma_0, \\ u(x) = f(x), & \forall\, x \in \Gamma_0, \end{cases} \tag{5.23}$$

where the new boundary is only $\Gamma_0$.

Hence, we have proven the following results:

**Theorem 5.4.** *Let $S$ be a subgraph of a directed weighted host graph $(G, \varpi)$, and let $\alpha$ be a nonnegative function defined on $G$ so that the expression* (5.1) *defines the operator $A$. Set $\Gamma = \partial S \cup \partial^* S$, $\overline{S} = S \cup \Gamma$ and let $\Gamma_0$ be a part of $\Gamma$ satisfying* (5.20) *which is reachable form $S$, i.e., for every $x$ in $S$ there exists a path originated at $x_0 = x$ and ending at some point $y$ in $\Gamma_0$. Given a function $f$ from $\overline{S}$ into $\mathbb{R}$, the discrete mixed boundary problem*

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S, \\ \partial_\nu u(x) = f(x), & \forall\, x \in \Gamma \smallsetminus \Gamma_0, \\ u(x) = 0, & \forall\, x \in \Gamma_0 \end{cases} \tag{5.24}$$

*has a unique solution $u$, which can be expressed by means of the Green function.* □

If $\Gamma_0$ is not reachable from $S$ then we need to decompose the subgraph $S$ as above, to get several Neumann type problems (and one mixed problem).

# 6 Probabilistic Interpretation I

In this section we discuss the problem in the full graph. To simplify the presentation, we take $\alpha = 0$.

## 6.1 Notation and Preliminaries

Our objective in this section is to give a probabilistic interpretation to the solution of the problem

$$Au = f, \tag{6.1}$$

posed in the full graph $G$. We know the solvability condition

$$(f, 1) = 0 \tag{6.2}$$

The probabilistic interpretation is linked to the fact that one can associate to the operator $A$ or to the weight $\varpi(x, y)$ a transition probability on the nodes of $G$, considered as the states of a Markov chain,

$$\pi(x, y) = \frac{\varpi(x, y)}{d_\varpi x}, \qquad d_\varpi x = \sum_y \varpi(x, y), \tag{6.3}$$

for every $x$ and $y$ in $G$. A very important, although immediate, observation is that there exists an invariant probability, associated to the transition probability $\pi(x, y)$, that is to say a probability $m(x)$, such that

$$\sum_x m(x)\pi(x, y) = m(y), \quad \forall\, y \in G \tag{6.4}$$

Indeed, calling

$$V(G) = \sum_x d_\varpi x,$$

we just take

$$m(x) = \frac{d_\varpi x}{V(G)} \tag{6.5}$$

Usually, the existence of the invariant probability, is a consequence of Ergodic Theory, requiring for instance the Doeblin's condition, see J.L. Doob[3]. Here, this condition is not satisfied, but the invariant probability is obtained trivially. As we shall see, most of the standard results, obtained from Ergodic Theory, except one, will be available. The next step is to introduce the operator

$$\Phi f(x) = \sum_y \pi(x,y) f(y). \tag{6.6}$$

It is convenient to consider a different norm than that used to define $L^2(G)$, namely

$$\|f\|_{L^\infty(G)} = \max_{\{x \in G\}} |f(x)|.$$

The notation $L^\infty(G)$ speaks for itself, although, of course, since we are dealing with a finite dimensional vector space, these two norms are equivalent. The interest of this norm is the fact that

$$\|\Phi\| \leq 1.$$

We have the relation

$$I - \Phi = d^{-1} A, \tag{6.7}$$

where the operator $I$ is the identity and $d$ is the multiplication by the function $d_\varpi$, i.e., $dv(x) = d_\varpi x \, v(x)$, for every $x$ in $G$. Note also

$$\Phi 1 = 1, \tag{6.8}$$

where by 1, we mean the constant function equal to 1, and

$$(m, \Phi v) = (m, v), \quad \forall \, v \in L^\infty(G), \tag{6.9}$$

which is the classical property of invariant probabilities. This amounts to

$$\Phi^* m = m, \tag{6.10}$$

where the operator $\Phi^*$ dual of $\Phi$ is given by

$$\Phi^* v(x) = \sum_y v(y) \frac{\varpi(x,y)}{d_\varpi y}. \tag{6.11}$$

Now equation (6.1) reads

$$(I - \Phi)u = d^{-1} f. \tag{6.12}$$

The solvability condition can be written as

$$(m, d^{-1} f) = \frac{(f,1)}{V(G)} = 0. \tag{6.13}$$

So for a general $f$, we can consider the usual ergodic formulation

$$(I - \Phi)u(x) + \rho = d^{-1} f(x), \tag{6.14}$$

where $\rho$ is a constant, necessarily given by

$$\rho = (m, d^{-1} f) = \frac{(f,1)}{V(G)}. \tag{6.15}$$

In terms of $A$, equation (6.14) reads

$$Au + \rho V(G) m = f, \tag{6.16}$$

and we know that $u$ is defined, up to a constant. As it is natural in the quotient space, we have so far chosen the constant, so that

$$(u, 1) = 0.$$

This has no probabilistic interpretation, so in the present context, we pick the constant so that

$$(u, m) = 0. \tag{6.17}$$

As seen in Section 5.2, the discrete maximum principle applied to $m(x)$ (with a symmetric weight $\varpi$) yields that $m$ is actually constant, i.e., $m(x) = C$ for every $x$. This reconciles the analytic and probabilistic interpretations.

## 6.2 Approximation

Here we have solved (6.14) directly. The usual approach is to consider the following approximation

$$(I - \alpha\Phi)u_\alpha(x) = d^{-1}f(x), \quad \forall\, x \in G, \tag{6.18}$$

where the $\alpha$ is a constant satisfying $0 < \alpha < 1$ destined to approach 1. Since $\alpha\Phi$ is a contraction operator in $L^\infty(G)$ with norm $\alpha$, the solution of (6.18) is trivially given by the series

$$u_\alpha = \sum_{\{n=0\}}^{\infty} \alpha^n \Phi^n d^{-1} f. \tag{6.19}$$

The usual results, consequence of Ergodic Theory, are summarized as follows:

**Theorem 6.1.** *The problem* (6.14) *has a solution* $(u, \rho)$, *where* $u$ *is unique up to an additive constant, i.e, unique if we impose* $(u, m) = 0$. *The constant* $\rho$ *is given by* (6.15) *and*

$$u(x) = \lim_{\alpha \to 1} \big(u_\alpha(x) - \frac{\rho}{1-\alpha}\big). \tag{6.20}$$

*Moreover, we have also*

$$(1-\alpha)u_\alpha(x) \to \rho, \quad \forall\, x \in G \tag{6.21}$$

*and*

$$\frac{1}{N}\sum_{n=0}^{N-1} \Phi^n d^{-1} f \to \rho, \;\; in \;\; L^\infty(G). \tag{6.22}$$

*Proof.* The uniqueness of $\rho$ is obvious, since there is an explicit formula. The uniqueness of $u$ follows from the fact that, if

$$(I - \Phi)u(x) = 0$$

then

$$Au = 0,$$

which yields $u$ is constant. Since $(m, u) = 0$, this constant is $u = 0$.

Consider next the function $u_\alpha$, solution of (6.18). Set

$$\tilde{u}_\alpha(x) = u_\alpha(x) - (m, u_\alpha) = u_\alpha(x) - \frac{\rho}{1-\alpha}.$$

It is easy to check from (6.18) that

$$(I - \alpha\Phi)\tilde{u}_\alpha + \rho = d^{-1}f. \tag{6.23}$$

From this, we deduce

$$1 - \alpha d\tilde{u}_\alpha + A\tilde{u}_\alpha + \rho m V(G) = f.$$

By taking the scalar product with $\tilde{u}_\alpha$, we get

$$1 - \alpha(d\tilde{u}_\alpha, \tilde{u}_\alpha) + (A\tilde{u}_\alpha, \tilde{u}_\alpha) = (f, \tilde{u}_\alpha).$$

From this energy equality and properties (3.3) and (3.6) of $A$, we deduce that

$$\|\tilde{u}_\alpha\|_{L^2(G)} \leq C.$$

Hence, property (6.21) follows immediately. Next we extract a subsequence, such that

$$\tilde{u}_\alpha \to u$$

and passing to the limit in (6.23) as $\alpha \to 1$, we deduce that $u$ is solution of (6.14), with $(m, u) = 0$. By the uniqueness of the limit, the full sequence converges, which is property (6.19). Next, from (6.14), one obtains

$$\Phi^n u - \Phi^{n+1} u + \rho = \Phi^n d^{-1} f.$$

Therefore, also

$$u - \Phi^N u + N\rho = \sum_{n=0}^{N-1} \Phi^n d^{-1} f.$$

Dividing by $N$, and letting $N$ tend to $\infty$, we obtain (6.22). The proof has been completed. □

*Remark* 6.2. On the other hand, the property

$$\|\Phi^n v - (m, v)\| \leq K\beta^n, \quad \beta < 1 \tag{6.24}$$

is not true, without additional assumptions. Therefore, from the fact

$$\sum_{n=0}^{\infty} \alpha^n \Phi^n (d^{-1}f - \rho)(x) \to u(x), \text{ as } \alpha \to 1 \tag{6.25}$$

we cannot deduce the formula

$$u(x) = \sum_{n=0}^{\infty} \Phi^n (d^{-1}f - \rho)(x) \tag{6.26}$$

which would be a consequence of (6.24). □

## 6.3 Limiting Average

Consider a probability space $(\Omega, \mathcal{A}, P)$, on which is constructed a Markov chain

$$y_0, y_1, \ldots, y_n, \ldots,$$

where $y_n = y_n(\omega)$ denotes the state of the Markov chain at time $n$. Clearly, $y_n$ are random variables with values in $G$. The states of the Markov chain are the nodes of $G$, so there are $N$ states. The Markov chain is governed by the transition probability $\pi(x, y)$ given by (6.3). Set

$$P(x, n, y) = \Phi^n \delta_y(x),$$

where

$$\delta_y(z) = \begin{cases} 1, & \text{if } z = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then, as it is well known

$$P\{y_n = y \,|\, y_0 = x\} = P(x, n, y).$$

Consider the filtration generated by the random variables $\{y_i\}$, i.e.,

$$\mathcal{F}^n = \sigma(y_0, y_1, \cdots, y_n) \tag{6.27}$$

then the following Markov property holds

$$P\{y_{n+k} = y \,|\, \mathcal{F}^k\} = P(y_k, n, y). \tag{6.28}$$

Next, consider a stopping time $\nu$ of $\mathcal{F}^n$, namely,

$$\{\nu \leq k\} \subset \mathcal{F}^k, \quad \forall\, k,$$

and the $\sigma$-algebra $\mathcal{F}^\nu$ defined by

$$\Gamma \subset \mathcal{F}^\nu \Longleftrightarrow \Gamma \cap \{\nu \leq k\} \subset \mathcal{F}^k.$$

Then one has also the strong Markov property, i.e.,

$$P\{y_{n+\nu} = y \,|\, \mathcal{F}^\nu\} = P(y_\nu, n, y). \tag{6.29}$$

Note that

$$\Phi^n f(x) = E\{f(y_n) \,|\, y_0 = x\}. \tag{6.30}$$

Therefore, we can give the probabilistic interpretation of $u_\alpha$,

$$u_\alpha(x) = E\{\sum_{n=0}^{\infty} \alpha^n d^{-1} f(y_n) \,|\, y_0 = x\}, \tag{6.31}$$

$$\rho = \lim_{N\to\infty} E\{\frac{1}{N} \sum_{n=0}^{N-1} d^{-1} f(y_n) \,|\, y_0 = x\} \tag{6.32}$$

and

$$u(x) = \lim_{\alpha\to 1} E\{\sum_{\{n=0\}}^{\infty} \alpha^n (d^{-1} f(y_n) - \rho) \,|\, y_0 = x\}, \tag{6.33}$$

but we cannot interchange the limits lim and the summation $\sum$ signs.

# 7 Probabilistic Interpretation II

In this section we discuss the Dirichlet problem in an induced and connected subgraph $S$ with a non-empty boundary $\partial S$ defined by (2.1). Again, to simplify the presentation, we take $\alpha = 0$.

## 7.1 Martingale Considerations

In a Markov chain the following *martingale* property holds, whose proof is left to the reader:

$$u(y_n) + \sum_{j=0}^{n-1} (u - \Phi u)(y_j) \text{ is a } \mathcal{F}^n \text{ martingale}, \quad \forall\, u, \tag{7.1}$$

recall that the operator $\Phi$ is defined by (6.6). Note that, from the general properties of martingales, we have also that for any stopping time $\nu$ of $\mathcal{F}^n$, the stopped martingale is also a martingale, hence

$$u(y_{n\wedge\nu}) + \sum_{\{j=0\}}^{n\wedge\nu-1} (u - \Phi u)(y_j) \text{ is a } \mathcal{F}^n \text{ martingale}, \quad \forall\, u \tag{7.2}$$

## 7.2 Exit Time

Consider an induced subgraph $S$ and its (non-empty) boundary $\partial S$. We define

$$\tau = \inf\{n \,:\, y_n \in G \setminus S\}. \tag{7.3}$$

It is a stopping time since

$$\tau > k \iff y_0 \in S, \cdots, y_k \in S$$

and this event belongs to $\mathcal{F}^k$. We are going to show

**Lemma 7.1.** *We have*

$$\tau < \infty, \quad a.s. \tag{7.4}$$

*and*

$$\text{if } y_0 = x \in S \text{ then } y_\tau \in \partial S. \tag{7.5}$$

*Proof.* Consider the solution $z$ of the following problem

$$\begin{cases} z(x) - \Phi z(x) = 1, & \forall\, x \in S, \\ z(x) = 0, & \forall\, x \in \partial S. \end{cases} \tag{7.6}$$

From the maximum principle, see Theorem 3.5, we have $z \geq 0$. We denote $P^x$ the probability on $(\Omega, \mathcal{A})$ with the property $y_0 = x$ a.s. Using the martingale property (7.2), with $\nu = \tau$, and noting that $j \leq n \wedge \tau - 1$ implies $y_j \in S$, taking account of (7.6), we get

$$E^x\big\{z(y_{n\wedge\tau}) + n \wedge \tau\big\} = z(x)$$

and from the positivity of $z$, it follows

$$E^x\big\{n \wedge \tau\big\} \leq z(x).$$

Letting $n$ tend to $\infty$, we deduce

$$E^x \tau \leq z(x).$$

Clearly, $E^x\{\cdot\}$ denote the expectation with respect to conditional probability $P^x$, i.e., $E^x\{\cdot \,|\, y_0 = x\}$. This proves (7.4).

Let us prove (7.5). We pick $z \in G - \bar{S}$. We have

$$\begin{aligned} E^x\big\{\mathbf{1}_{\{y_\tau = z\}}\big\} &= \sum_{n=0}^{\infty} E^x\big\{\mathbf{1}_{\{y_\tau = z\}}\mathbf{1}_{\{\tau = n\}}\big\} = \\ &= \sum_{n=0}^{\infty} E^x\big\{\mathbf{1}_{\{y_n = z\}}\mathbf{1}_{\{y_1 \in S, \cdots, y_{n-1} \in S\}}\big\} = \\ &= \sum_{n=0}^{\infty} E^x\big\{\mathbf{1}_{\{y_1 \in S, \cdots, y_{n-1} \in S\}}\pi(y_{n-1}, z)\big\}. \end{aligned}$$

But $\pi(y_{n-1}, z) = 0$, since $y_{n-1} \in S, z \in G - \bar{S}$. Therefore, we have

$$E^x\big\{\mathbf{1}_{\{y_\tau = z\}}\big\} = 0.$$

Since $z$ is any node in $G \setminus \bar{S}$, and there are a finite number of such nodes, we get

$$E^x\big\{\mathbf{1}_{\{y_\tau \in G - \bar{S}\}}\big\} = 0.$$

So $y_\tau$ belongs to $\bar{S}$ almost surely. Since $y_\tau$ is not in $S$, the property (7.5) is proven. □

### 7.3 Explicit Formula

We consider here the nonhomogeneous Dirichlet problem

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S, \\ u(x) = g(x), & \forall\, x \in \partial S. \end{cases} \tag{7.7}$$

Our objective is to prove the following result

**Theorem 7.2.** *The solution of problem* (7.7) *is given by*

$$u(x) = E^x\Big\{ \sum_{\{j=0\}}^{\tau-1} d^{-1} f(y_j) + g(y_\tau) \Big\}. \tag{7.8}$$

*Proof.* Note first that (7.6) is equivalent to

$$\begin{cases} (I - \Phi)u(x) = d^{-1} f(x), & \forall\, x \in S, \\ u(x) = g(x), & \forall\, x \in \partial S. \end{cases}$$

From the martingale property, see (7.2), we have

$$u(x) = E^x\Big\{ \sum_{\{j=0\}}^{n\wedge\tau-1} (u - \Phi u)(y_j) + u(y_{n\wedge\tau}) \Big\}.$$

Since, $y_j \in S$ for every $j \leq n \wedge \tau - 1$, using the equation (7.8) we get

$$u(x) = E^x\Big\{ \sum_{\{j=0\}}^{n\wedge\tau-1} d^{-1} f(y_j) + u(y_{n\wedge\tau}) \Big\}.$$

Letting $n$ tend to $\infty$, the result (7.8) follows immediately, after making use of Lemma 7.1. □

# 8 Probabilistic Interpretation III

In this section we discuss the Neumann problem in an induced and connected subgraph $S$. Again, to simplify the presentation, we take $\alpha = 0$.

## 8.1 Setting and Notation

We consider here the problem (3.32)

$$\begin{cases} Au(x) = f(x), & \forall\, x \in S \\ \dfrac{\partial u}{\partial \nu_A}(x) = g(x), & \forall\, x \in \partial S \end{cases} \tag{8.1}$$

We recall the solvability condition, see (3.36)

$$\int_S f + \int_{\partial S} g = 0. \tag{8.2}$$

We recall the operator $A_{\bar{S}}$, so that the problem is equivalent to

$$\begin{cases} A_{\bar{S}}u(x) = f(x), & \forall\, x \in S, \\ A_{\bar{S}}u(x) = g(x), & \forall\, x \in \partial S. \end{cases} \tag{8.3}$$

Next, define

$$d_{\varpi,\bar{S}}(x) = \sum_{y \in \bar{S}} \varpi(x,y), \quad \forall\, x \in \bar{S}, \tag{8.4}$$

$$d'_{\varpi}(x) = \sum_{y \in S} \varpi(x,y), \quad \forall\, x \in S \tag{8.5}$$

and

$$d_{\bar{S}}v(x) = \begin{cases} d_{\varpi,\bar{S}}(x)v(x), & \forall\, x \in S, \\ d'_{\varpi}(x)v(x), & \forall\, x \in \partial S, \end{cases} \tag{8.6}$$

$$\Phi_{\bar{S}} = I - d_{\bar{S}}^{-1} A_{\bar{S}}. \tag{8.7}$$

## 8.2 Statement of Results

We are, in a situation similar to the case of a full graph. So we give the results without details. We introduce the invariant probability

$$m_{\bar{S}}(x) = \begin{cases} \dfrac{d_{\varpi,\bar{S}}(x)}{V(\bar{S})}, & \forall\, x \in S, \\ \dfrac{d'_{\varpi}(x)}{V(\bar{S})}, & \forall\, x \in \partial S, \end{cases} \tag{8.8}$$

with

$$V(\bar{S}) = \int_S d_{\varpi,\bar{S}} + \int_{\partial S} d'_{\varpi}.$$

If, we consider general $f, g$, without compatibility conditions, then the equation to be solved is written as follows

$$\begin{cases} Au(x) + \rho V(\bar{S}) d_{\varpi,\bar{S}}(x) = f(x), \quad \forall\, x \in S, \\ \dfrac{\partial u}{\partial \nu_A}(x) + \rho V(\bar{S}) d'_{\varpi}(x) = g(x), \quad \forall\, x \in \partial S, \end{cases} \tag{8.9}$$

and

$$\rho = \frac{\displaystyle\int_S f + \int_{\partial S} g}{V(\bar{S})} \tag{8.10}$$

We associate with this problem the following equation

$$(I - \Phi_{\bar{S}})u + \rho = D_{\bar{S}}^{-1} L \tag{8.11}$$

with

$$\ell(x) = \begin{cases} f(x), \text{ if } x \in S, \\ g(x), \text{ if } x \in \partial S, \end{cases} \tag{8.12}$$

and (8.11) means, of course,

$$A_{\bar{S}} u + \rho d_{\bar{S}} = \ell. \tag{8.13}$$

We next introduce the approximation

$$(I - \alpha \Phi_{\bar{S}}) u_\alpha(x) = d_{\bar{S}}^{-1} \ell(x), \quad \forall\, x \in \bar{S}, \tag{8.14}$$

where $0 < \alpha < 1$, and we let $\alpha$ tend to 1. We have, as for (6.19)

$$u_\alpha = \sum_{n=0}^{\infty} \alpha^n \Phi_{\bar{S}}^n d_{\bar{S}}^{-1} \ell(x) \tag{8.15}$$

and we state the equivalent of Theorem 6.1.

**Theorem 8.1.** *The solution of* (8.11), $(u, \rho)$, *unique if we impose* $(u, m_{\bar{S}}) = 0$, *and* $\rho$ *is given by* (8.10), *and*

$$u(x) = \lim_{\alpha \to 1}(u_\alpha(x) - \frac{\rho}{1-\alpha}). \tag{8.16}$$

*We have also*

$$(1-\alpha)u_\alpha(x) \to \rho, \quad \forall\, x, \tag{8.17}$$

*and*

$$\frac{1}{N}\sum_{n=0}^{N-1} \Phi^n d_{\bar{S}}^{-1}\ell \to \rho, \quad in \;\; L^\infty(G). \qquad \square \tag{8.18}$$

It remains to state the probabilistic interpretation. We have

$$u_\alpha(x) = E^x\big\{\sum_{n=0}^{\infty} \alpha^n (\frac{f(y_n)}{d_{\varpi,\bar{S}}(y_n)}\mathbf{1}_{\{y_n \in S\}} + \frac{g(y_n)}{d'_{\varpi}(y_n)}\mathbf{1}_{\{y_n \in \partial S\}})\big\} \tag{8.19}$$

and

$$\rho = \lim_{N\to\infty} E^x\big\{\frac{1}{N}\sum_{n=0}^{N-1}(\frac{f(y_n)}{d_{\varpi,\bar{S}}(y_n)}\mathbf{1}_{\{y_n \in S\}} + \frac{g(y_n)}{d'_{\varpi}(y_n)}\mathbf{1}_{\{y_n \in \partial S\}})\big\}, \tag{8.20}$$

$$u(x) = \lim_{\alpha\to 1} E^x\big\{\sum_{n=0}^{\infty} \alpha^n (\frac{f(y_n)}{d_{\varpi,\bar{S}}(y_n)}\mathbf{1}_{\{y_n \in S\}} + \frac{g(y_n)}{d'_{\varpi}(y_n)}\mathbf{1}_{\{y_n \in \partial S\}} - \rho)\big\}. \tag{8.21}$$

# 9 Conclusions

Directed and undirected weighted graphs have been considered. Discrete versions of the stationary Neumann, Dirichlet and mixed problems have been successfully studied, in variational and non-variational forms. Probabilistic interpretations of the solutions are given by means of stationary Markov chain.

Several extensions are in order, first the time-dependent case has to be discussed. Next, several non-linear problems should be studied, e.g., where the weight of the graph is depending on a parameter which can be changed. This will involve some optimization and the application of the dynamic programming principle. All this is will be part of future works.

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